1 “Greeks”: Hedging with Options

- Option Market Makers – Buy and sell options to investing public. Market makers earn bid-ask spread on options written. Market makers do not want to assume risk, so they will want to hedge their portfolios. If investing public is net buy of options, market makers must be net sellers. How can they hedge?

- Delta Hedging
- Theta
- Gamma and Delta-Gamma Hedging
- Vega
- Managing Delta, Gamma and Vega
- Hedging in Practice

- Black-Scholes Formula

$$c_0 = S_o e^{-qT} \mathrm{N}(d_1) - \mathrm{K}\, e^{-rT} \mathrm{N}(d_2)$$

$$d_1 = [\ln(S_0/K) + (r - q + s^2/2)\mathrm{T}] / (s\ \mathrm{T}^{1/2}); \quad d_2 = d_1 - (s\ \mathrm{T}^{1/2})$$

2

Sensitivity of Option Prices II

$$c_0 = S_0\, e^{-qT}\, \mathrm{N}(d_1) - K\, e^{-rT}\, \mathrm{N}(d_2) \quad \text{and}$$

$$p_0 = -S_0 e^{-qT}\, \mathrm{N}(-d_1) + K\, e^{-rT}\, \mathrm{N}(-d_2)$$

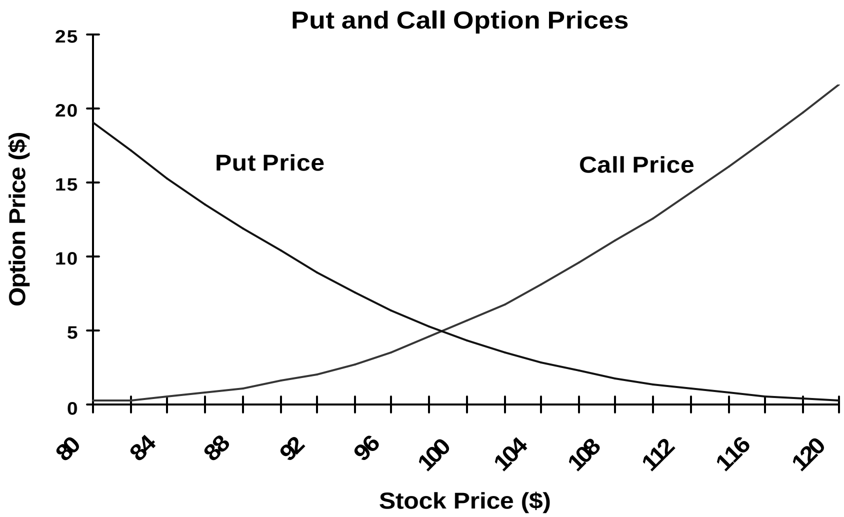
$$d_1 = [\ln(S_0/K) + (r - q + s^2/2)\mathrm{T}] \,/\, (s\ \mathrm{T}^{1/2}); \quad d_2 = d_1 - (s\ \mathrm{T}^{1/2})$$

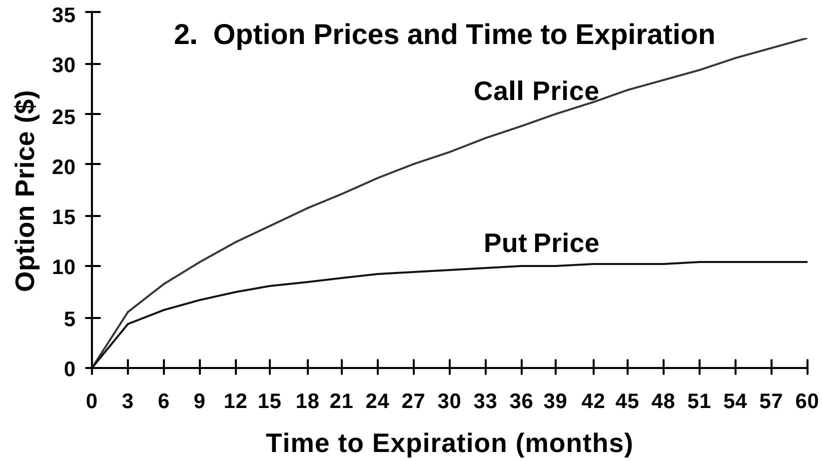
- Theta (Θ) – Theta in a delta-neutral portfolio is a proxy for gamma.

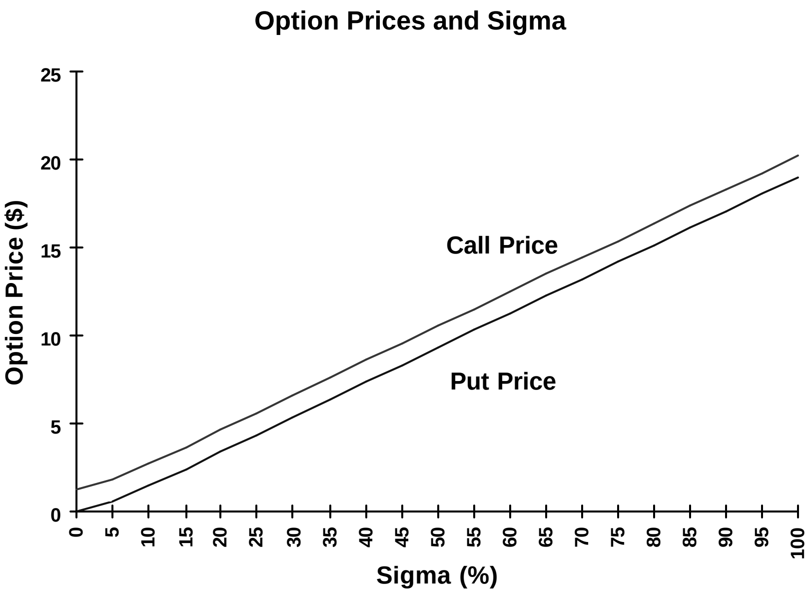
Change in Option Price wrt:	**Greek**	**Call**	**Put**	**Formula Euro Call/Put**
Stock Price ($) $\partial c/\partial S$	Delta Δ	+	–	$e^{-qT}\,\mathrm{N}(d_1)$ / $-e^{-qT}\,\mathrm{N}(-d_1)$
Stock Price (%)	Eta	>1	<1	$e^{-qT}\,\mathrm{N}(d_1){*}\mathrm{S}_0/\mathrm{c}_0$
Time until expiration $\partial f/\partial t$	Theta Θ	+/?	+/?	$\mathrm{S}_0 N'(d_1)\, s\ e^{-qT}/(2\mathrm{T}^{1/2}) - rKe^{-rT}N(d2)$
Volatility (s) (%) $\partial f/\partial s$	Vega ν	+	+	$\mathrm{S}_0\ e^{-qT}\, n(d_1)\ \mathrm{T}^{1/2}$
Risk-free rate (r) $\partial f/\partial r$	Rho ρ	+	–	
Dividend Yield (q) $\partial f/\partial q$		–	+	
Change in Delta wrt Asset Price	Gamma Γ	+	+	$\mathrm{N}'(d_1)e^{-qT} / S_0\, s\, T^{1/2}$ for both
$(\partial^2 f/\partial S^2)$				$\partial\Pi \approx \Theta\partial t + \tfrac{1}{2}\Gamma\partial S^2$ *
				*for delta neutral portfolio

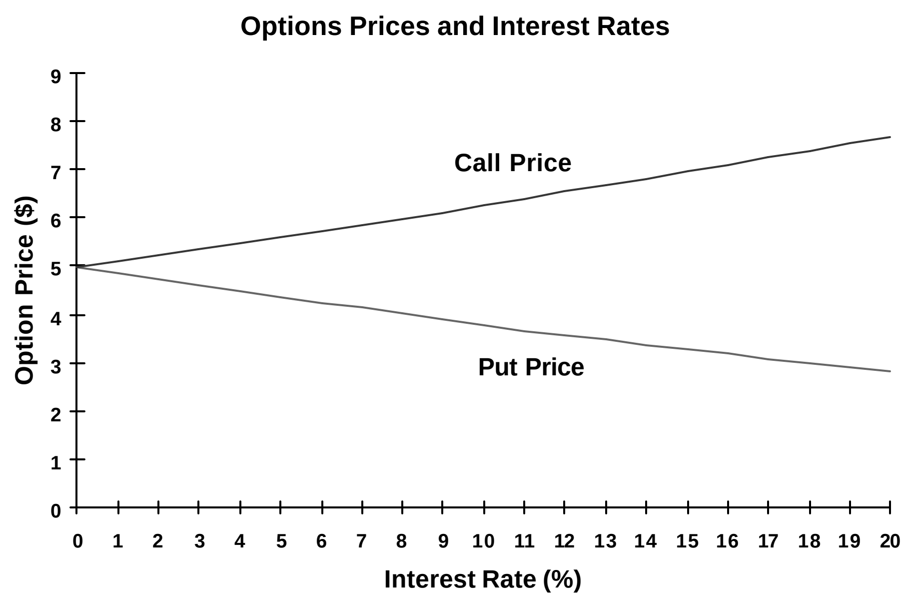
3

Sensitivity of Option Prices III









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Delta Hedging

- Suppose a bank sold for $300,000 a European call option on 100,000 shares of a non-dividend paying stock. How would bank hedge its risk?
  - S_0=49; K=50; r=5%; σ=20%; T = 140 days (0.3836 yrs).
  - Black-Scholes value of call option portfolio is ≈ $239,600 ($c$=2.39600).

- Hedging possibility I: Naked position
  –

- Hedging possibility II: Covered position
  –

- Hedging possibility III: Stop-loss strategy:
  –
  –

- Hedging possibility IV: Delta Hedging. $\Delta_{call} = 0.5214$
  –

5 Delta Hedging

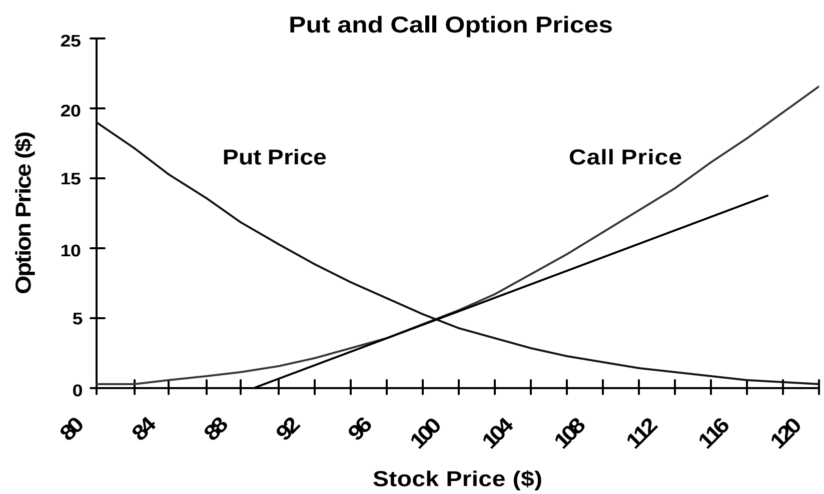
- Hedging a portfolio – add traded asset to existing portfolio so that $\partial P/\partial S = 0$.
  - value of portfolio: $P = NP + N_T P_T$.
  - value of current position is NP .
  - value of asset position being added is $N_T P_T$.

- We want to find number of assets N_T^* to purchase such that $\partial P/\partial S = 0$.
  - Recall hedge is effective only for very small change in S.

$$\frac{\partial \Pi}{\partial S} = N\frac{\partial P}{\partial S} + N_T{}^* \frac{\partial P_T}{\partial S} = 0 \quad \text{or}$$

$$\frac{\partial \Pi}{\partial S} = N\Delta + N_T{}^* \Delta_T = 0$$

-



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Delta Hedging Intuition

$$N_T* = -\frac{N\Delta}{\Delta_T} = -\frac{\Delta_p}{\Delta_T}$$

- Q: You are long calls. How can position be hedged with underlying asset?
- A:

- Q: You are short calls. How can position be hedged with underlying asset?
- A:

- Q: You are long puts. How can position be hedged with underlying asset?
- A:

- Q: You are short puts. How can position be hedged with underlying asset?
- A:

- Q: The delta of an option position is –52,140. How can this be hedged?
- A:

7 # Delta Hedging Example I

- Q1: Suppose a bank sold for $300,000 European call options on 100,000 shares of a non-dividend paying stock. How would bank hedge its risk with shares?
  - S_0=49; K=50; r=5%; σ=20%; T=0.3836 yrs; c=2.39600. Δ_{call}=0.5214.
  - Q1&2: Create/interpret delta-neutral portfolio using underlying shares.
  - Q3: Create a delta-neutral portfolios using Euro puts (same features)
  - Q4: Create a delta-neutral portfolios using Euro calls (T=1).

- A1: Create *delta neutral portfolio* using shares.
  - Recall
  - N_T* =
  - New portfolio Δ_p* =

- A2: Interpretation: Hedging holds for small changes in stock price.
  - Suppose S_t=$50. This implies new value of call by B-S is c*=
  - Orig Π_p =
  - Then Π_p* =
  - Change in value of portfolio: Π_p*– Π_p =

8 # Delta Hedging Example II

- Q1: Suppose a bank sold for \$300,000 European call options on 100,000 shares of a non-dividend paying stock. How would bank hedge its risk with shares?
  - S_0=49; K=50; r=5%; σ=20%; T=0.3836 yrs; c=2.39600.
  - Q1&2: Create/interpret a delta-neutral portfolio using underlying shares.
  - Q3: Create a delta-neutral portfolios using Euro puts (same features)
  - Q4: Create a delta-neutral portfolios using Euro calls (T=1).

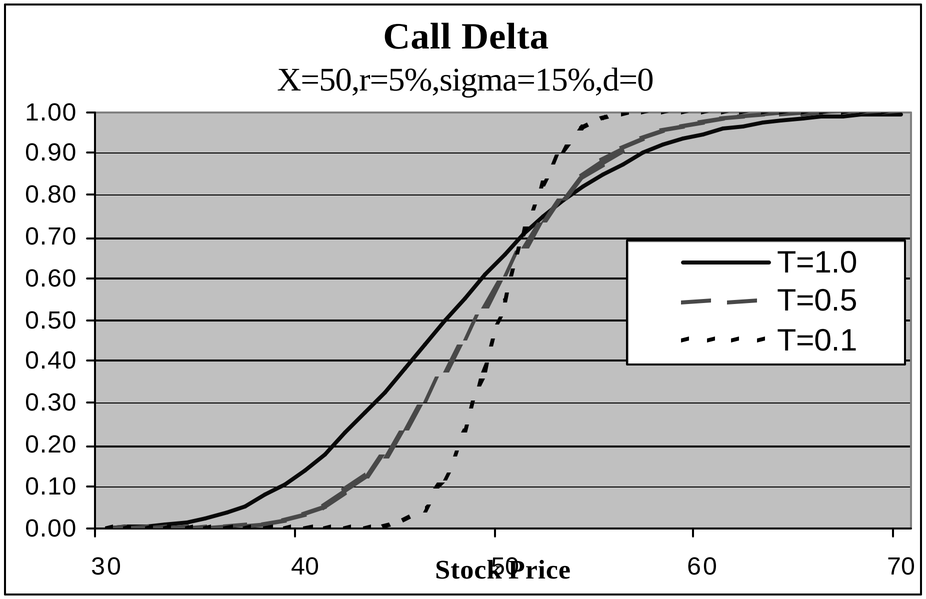
- A3: Hedging position with Euro put (same features). Suppose $\Delta_{put} = -0.4784$.
  - $N_T^* = -\Delta_p/\Delta_T =$

- A4: Hedging position with Euro call (T=1) with $\Delta_{call} = 0.5983$.
  - $N_T^* = -\Delta_p/\Delta_T =$

9

Delta Hedging Analysis

- Rebalancing - hedge position must be frequently rebalanced.
  - Consider previous example. At S_t=$50, new delta is 0.5859 (vs 0.5214).
  - Hedger must purchase additional 6,430 shares to maintain hedge.
  - Delta hedging with shares on written option involves "buy high, sell low"!
  -
- Delta hedging portfolios
  - Delta of portfolio is weighted avg of security deltas.
  - Just one trade (e.g., index-linked security) may be required to hedge.
  - Traders may rebalance daily.
- Delta vs stock price
  - $\Delta \approx 0$ if far out-of-money
  - For long call $0<\Delta<1$.
  - For long put $-1<\Delta<0$.
  - Delta may increase or decrease over time.
-



Delta Hedging I

- Delta Hedging –
  Near exp, option is in-the-money (delta=1), so position is fully hedged.
- Cost of hedge –
- Rebalancing –
  Often improves hedge.
  ....vs transaction costs.

Wk	Stock Price	Delta	Shares Bought	Cost of Shares	Cumm Cost	Interest Cost
0	49.00	0.521	52,100	2,552,900	2,552,900	2,449
1	48.12	0.458	(6,400)	(307,968)	2,247,381	2,156
2	47.37	0.400	(5,800)	(274,746)	1,974,791	1,895
3	50.25	0.596	19,700	989,925	2,966,611	2,846
4	51.75	0.693	9,700	501,975	3,471,432	3,330
5	53.12	0.774	8,000	424,960	3,899,722	3,741
6	53.00	0.771	(200)	(10,600)	3,892,863	3,735
7	51.87	0.706	(6,500)	(337,155)	3,559,443	3,415
8	51.38	0.674	(3,200)	(164,416)	3,398,442	3,260
9	53.00	0.787	11,200	593,600	3,995,302	3,833
10	49.88	0.550	(23,600)	(1,177,168)	2,821,967	2,707
11	48.50	0.412	(13,800)	(669,300)	2,155,375	2,068
12	49.88	0.542	13,000	648,440	2,805,882	2,692
13	50.37	0.591	4,800	241,776	3,050,350	2,926
14	52.13	0.768	17,800	927,914	3,981,191	3,819
15	51.88	0.759	(900)	(46,692)	3,938,318	3,778
16	52.87	0.865	10,600	560,422	4,502,518	4,320
17	54.87	0.978	11,300	620,031	5,126,869	4,919
18	54.62	0.990	1,100	60,082	5,191,869	4,981
19	55.87	1.000	1,000	55,870	5,252,720	5,039
20	57.25	1.000	0.00	0.00	5,257,759	
Total			**99,900**			

11

Delta Hedging II

- Delta Hedging
  Near exp, option is out-of-money (delta=0), so position is not hedged.

- Cost of hedge –

Wk	Stock Price	Delta	Shares Bought	Cost of Shares	Cumm Cost	Interest Cost
0	49.00	0.521	52,100	2,552,900	2,552,900	2,449
1	49.75	0.567	4,600	228,850	2,784,199	2,671
2	52.00	0.705	13,800	717,600	3,504,470	3,362
3	50.00	0.579	(12,600)	(630,000)	2,877,832	2,761
4	48.38	0.459	(12,100)	(585,398)	2,295,195	2,202
5	48.25	0.443	(1,600)	(77,200)	2,220,197	2,130
6	48.75	0.475	3,200	156,000	2,378,327	2,282
7	49.63	0.540	6,500	322,595	2,703,204	2,593
8	48.25	0.419	(12,000)	(579,000)	2,126,797	2,040
9	48.25	0.410	(900)	(43,425)	2,085,412	2,001
10	51.12	0.658	24,700	1,262,664	3,350,077	3,214
11	51.50	0.692	3,400	175,100	3,528,391	3,385
12	49.88	0.542	(14,900)	(743,212)	2,788,564	2,675
13	49.88	0.538	(400)	(19,952)	2,771,287	2,659
14	48.75	0.400	(13,900)	(677,625)	2,096,321	2,011
15	47.50	0.236	(16,400)	(779,000)	1,319,332	1,266
16	48.00	0.261	2,600	124,800	1,445,398	1,387
17	46.25	0.062	(20,000)	(925,000)	521,785	501
18	48.13	0.183	12,100	582,373	1,104,658	1,060
19	46.63	0.007	(17,600)	(820,688)	285,030	273
20	48.12	0.000	(600)	(28,872)	256,431	
Total			**0**			

12 # Theta

- Theta (Θ) – change in option price with respect to change (passage) in time.
  - Sometimes called "time decay". For share of stock, Θ=0.
  - Generally Θ<0 on Euro options; may be Θ>0 for Euro put on non-div stock
  - Usually quoted "per calendar day" (Θ/365) or "per trading day" (Θ/252).
  - Not hedge parameter, but useful for estimating gamma (later).
  -
- Theta on Euro put - if put is deep-in-money Θ>0.
  - Put may be worth less than intrinsic value, due to opp cost from not exercising and earning interest on proceeds.
  -
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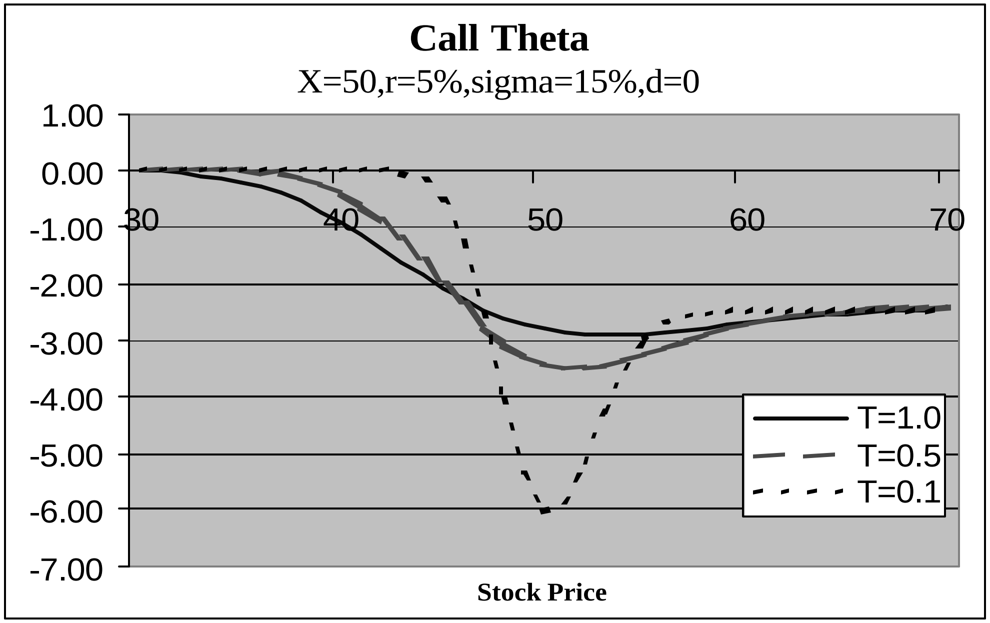


13 # Theta Example

- Suppose a bank sold for $300,000 European call options on 100,000 shares of a stock. A delta-neutral portfolio is formed by buying 52,140 shares (52,140*49 = $2,554,860). S_0=49; K=50; r=5%; σ=20%; T=0.3836 yr; c=2.39600.
  - Π_p = –$239,600 + $2,554,860 = $2,315,260.
  - Calculate and interpret theta on portfolio.

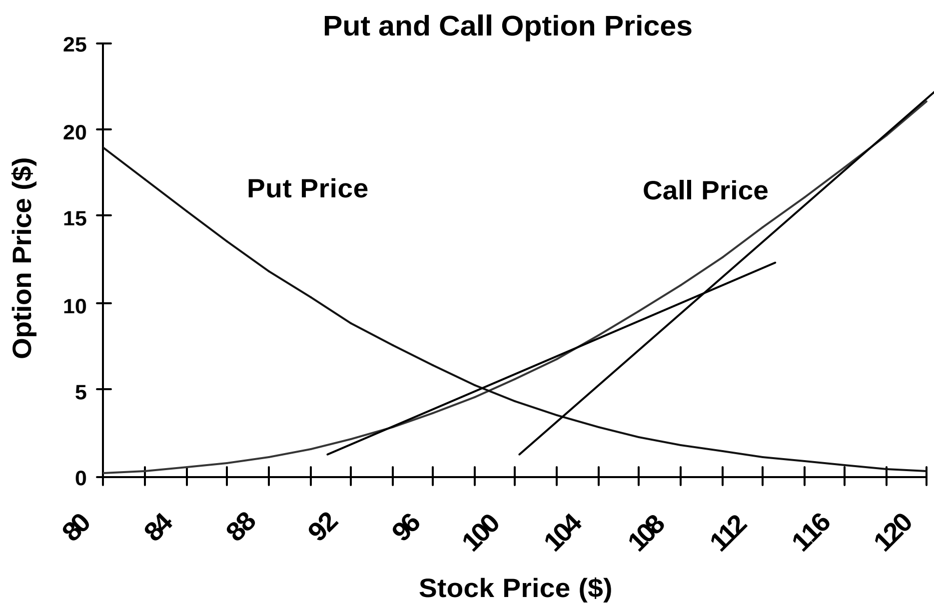
- A1: Calculate and interpret theta on portfolio.
  - Note
  - Portfolio: Θ_p =
  -

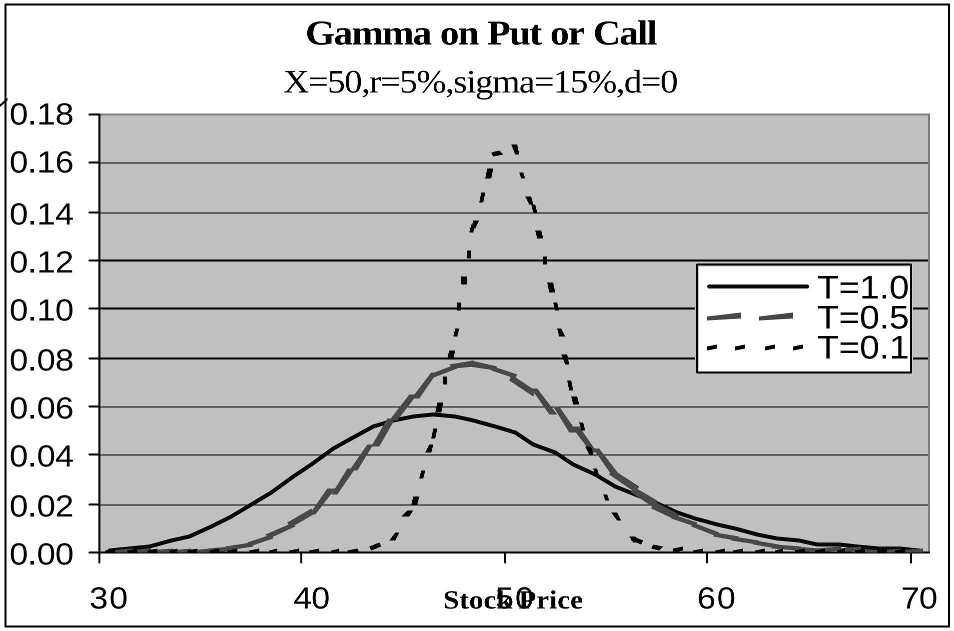
- A2: Actual change in value of portfolio after one day
  - After one day T_1=0.3808; c_1=2.38417;
  - Π_p* =
  - Change in Π_p = Π_p*–Π_p =

14

Gamma

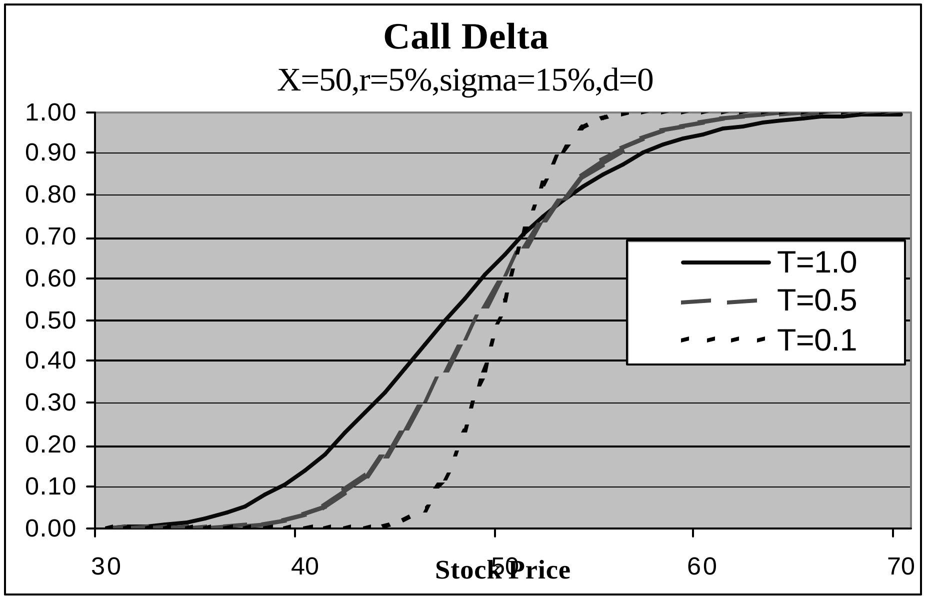
- Gamma (Γ) – rate of change of delta (Δ) with respect to change in price of underlying asset (*curvature* of delta curve.) Useful for delta-neutral portfolios.
  - Used to measure effect of changing stock prices on delta-neutrality.
  - For small gamma, delta neutrality maintained with less rebalancing.
  - For Euro calls and puts,
  - For deep out-of-money options,
  - For delta-neutral portfolio, profit sensitivity is: $\partial\Pi \approx \Theta\partial t + \frac{1}{2}\Gamma\partial S^2$.
  -

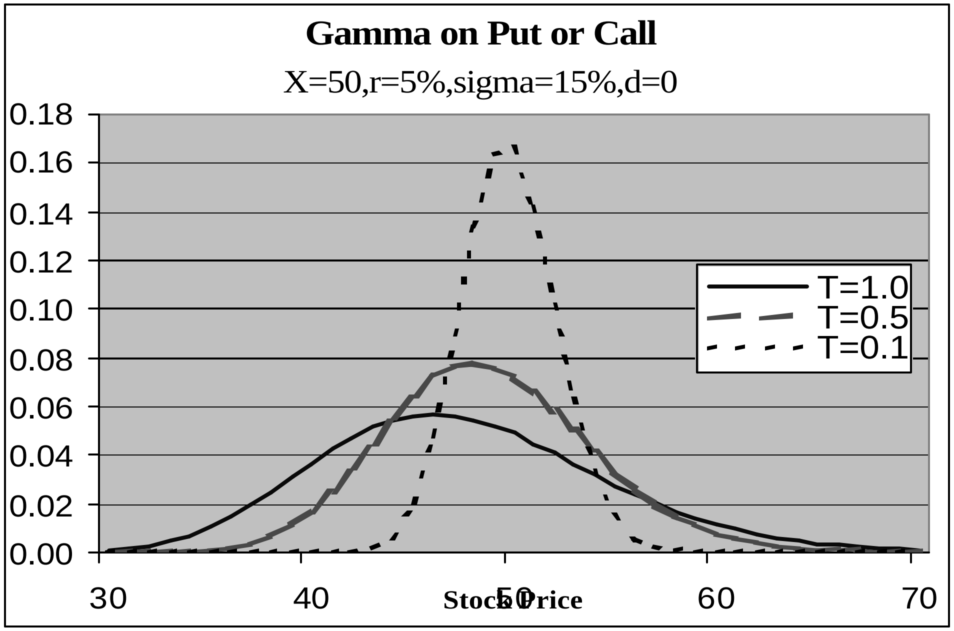




15 Gamma and delta relationship

- Option far out-of-money
  - –
- Option far in-the-money
  - –
- Option near at-the-money
  - –





16 # Gamma Example

- Suppose a bank sold for $300,000 European call options on 100,000 shares of a stock. A delta-neutral portfolio is formed by buying 52,140 shares ($2,554,860). S_0=49; K=50; r=5%; σ=20%; T=0.3836 yr; c=2.39600; Δ_{call}=0.5216
  – Calculate and interpret gamma on portfolio.

- A1: Calculate and interpret gamma.
  – Note that Γ_{share}=0; Γ_{call}=0.06564.
  – So Γ_p =
  –

- A2: Actual change in delta of portfolio as stock rises to $50.
  – Old Δ_p = –100,000*0.5216 + 52,140*1 = 0.
  – New Δ_p* = –100,000*0.5859 + 52,140*1 = –6,430.
  – Change in delta: Δ_p*– Δ_p= –6,430 – 0 = –6,430.
  – Since Δ_p drops, purchase more shares to again make portfolio delta-neutral.

- A3: Find change in dollar value of delta-neutral portfolio.
  – Recall that for delta-neutral portfolio: $\partial\Pi \approx \Theta\partial t + ½\Gamma\partial S^2$.
  – If stock jumps to S_t=$50, then $\partial\Pi \approx \Theta(0) + ½(–6,564)(1^2) = –\$3,282$.

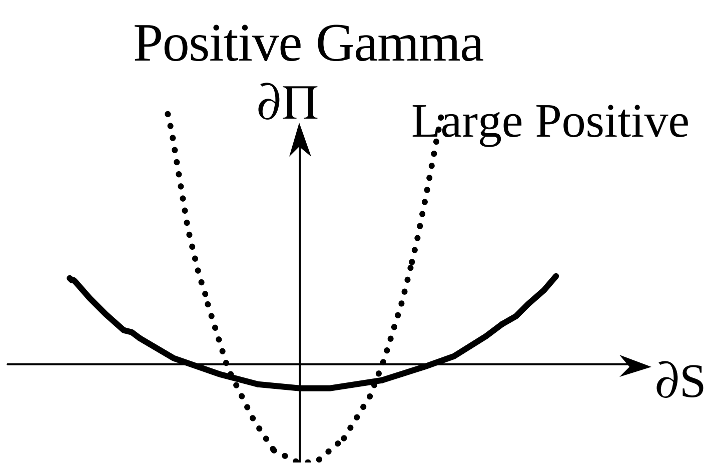
17 # Gamma on Delta-Neutral Portfolio's

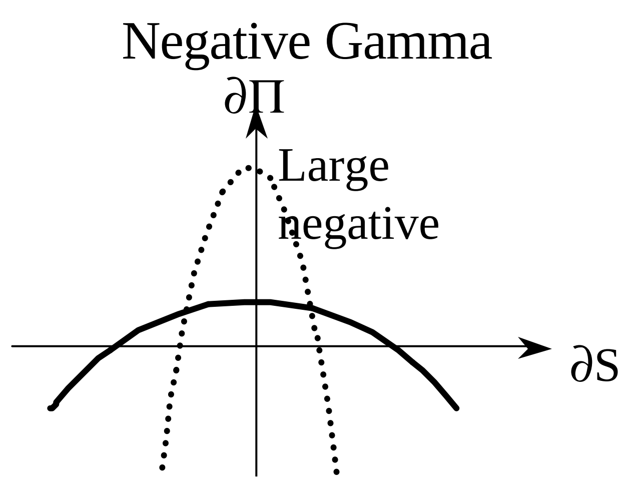
- Suppose gamma is positive on delta-neutral portfolio, so $\partial\Pi \approx \Theta\partial t + \frac{1}{2}\Gamma\partial S^2$
  - Portfolio declines if $\partial S=0$ (or ∂S small), since $\Theta<0$.
  - For written options, $\Gamma<0$, so time decay reduces amount to close position.

- Effect of large ∂S when gamma is positive on delta-neutral portfolio.
  - Suppose $\Gamma_{port}>0$; then $\partial\Delta/\partial S>0$; Consider portfolio of long call; short share;
  - $\uparrow S \Rightarrow \uparrow$calls and $\downarrow$Short shares. Since Δ_{call} increases, $\uparrow$calls > $\downarrow$Short shares

- Gamma-neutrality – immunizes portfolio to larger changes in asset price.
  - Option (or other non-linear payoff) is req'd to change a portfolio's gamma.

Effects on Value of Delta-Neutral Portfolio of Change in Stock Price





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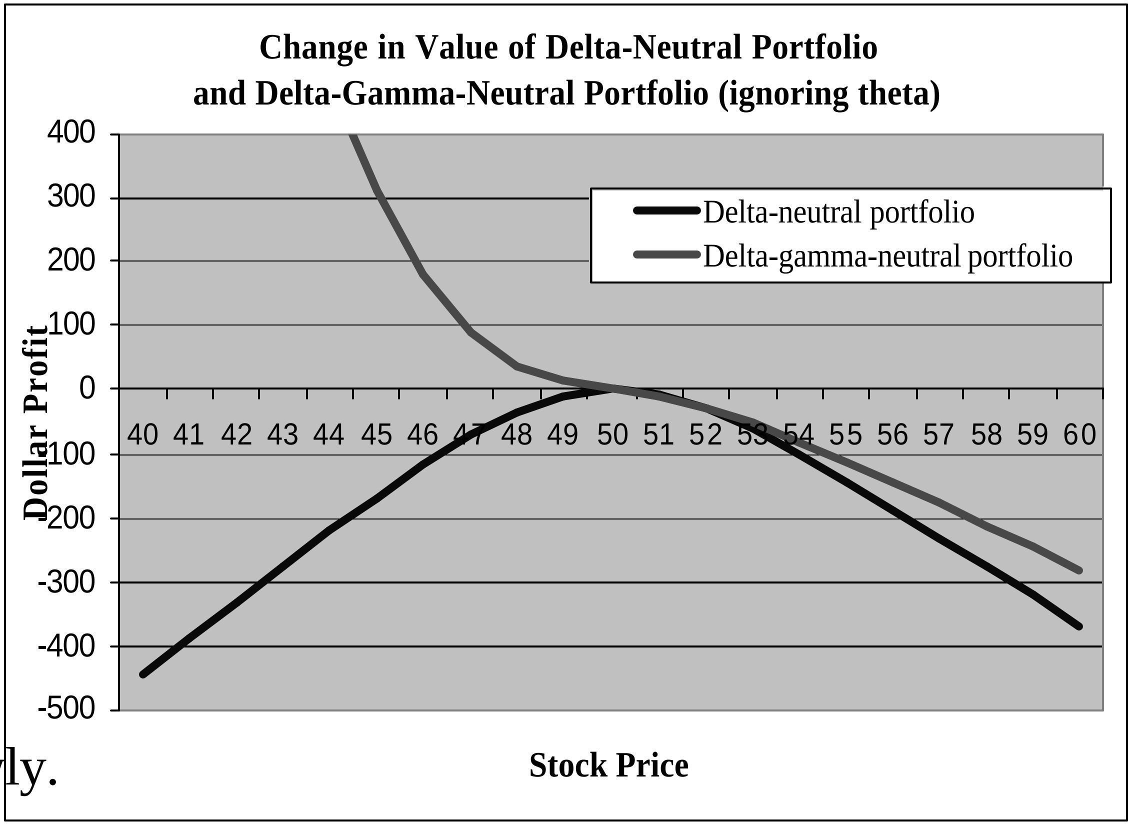
Gamma Neutrality

- Gamma-neutral portfolios - Making a delta-neutral portfolio also gamma neutral immunizes portfolio to larger changes in asset (stock) price.
  - Adjust portfolio gamma with options (or other non-linear payoff).
  -
- Constructing gamma neutral portfolio with traded options (with Γ_T)
  - Γ-neutrality on portfolio (Γ_p) using options (Γ_T): $N_T^* = -\Gamma_p/\Gamma_T$
- Q: Suppose a bank sold for $300,000 European call options on 100,000 shares of a stock. A delta-neutral portfolio is formed by buying 52,140 shares ($2,554,860). S_0=49; K=50; r=5%; σ=20%; T=0.3836 yr; c=2.39600; Δ_{call}=0.5216; Γ_p=–6,564.
  - Suppose put has Γ_T=0.0629 and Δ_T=–0.4142.
  - How can portfolio be made gamma and delta neutral?
- A1: Gamma neutral with puts: $N_T^* = -\Gamma_p/\Gamma_T =$
- A2: Delta neutrality: New $\Delta_p^* =$
  - Delta neutral with shares: $N_T^* = -\Delta_p/\Delta_T =$.

19 # Delta-Gamma Neutrality

- Delta-Gamma neutral portfolios have better risk profiles.
  - Plot below shows changing portfolio value for
    - Delta-neutral portfolio (short 100 call; long 55 shares) ($\Gamma<0$).
    - Delta-Γ-neutral portfolio (short 100 call; long 277 put; long 66 shares)
    - Delta-Γ neutral looks much better!
  - Why not always go for delta-gamma neutrality?
    -
    -

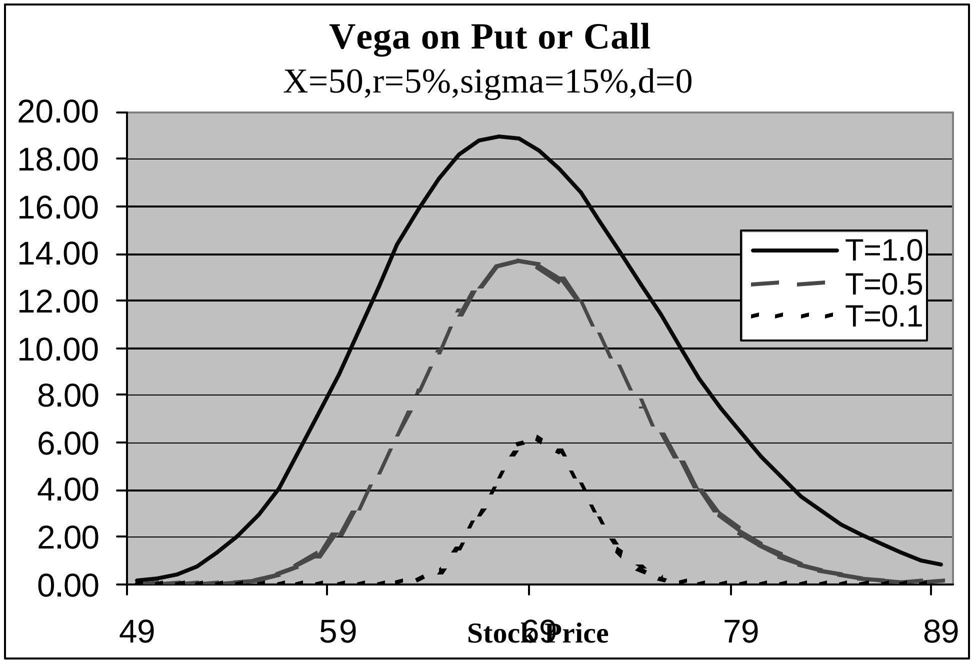


- Note: Δ-neutral portfolio:
  short call ($0<\Delta_{call}<1$);
  long asset ($\Delta_{share}=1$)
  - $S\downarrow \Rightarrow$ short call $\uparrow$ slowly.
  - $S\uparrow \Rightarrow$ short call $\downarrow$ slowly.

- Note: Δ-Γ-neutral portfolio:
  long put ($-1<\Delta_{put}<0$)
  - $S\downarrow \Rightarrow$ put+asset $\uparrow$ quickly.
  - $S\uparrow \Rightarrow$ put+short call $\downarrow$ slowly.

20 # Vega

- Vega (ν) – change in value of portfolio with respect to change in volatility.
  - Volatility of financial assets changes over time (shortcoming of B-S).
  - Vega can be used to estimate change in value of option w.r.t volatility.
  -
- Vega neutrality – It is possible to make a portfolio vega neutral.
  - Gamma neutrality does not ensure vega neutrality.
  -
- Delta-Gamma-Vega neutrality
  -



21

Vega Example

- Q: Suppose a bank sold for \$300,000 European call options on 100,000 shares of a stock. A delta-neutral portfolio is formed by buying 52,140 shares (\$2,554,860). S_0=49; K=50; r=5%; σ=20%; T=0.3836 yr; c=2.39600; Δ_{call}=0.5216; Θ_p=1,180; Γ_p=–6,564.
  - Π_p = – \$240,053 + \$2,554,860 = \$2,314,807.
  - Calculate and interpret hedged portfolio's vega.

- A1: Calculate portfolio's vega
  - Note that: ν_{call}=0.12089; and ν_{share}=0;
  - So ν_p=
  -

- A2: Interpret portfolio's vega
  - Suppose volatility increases by 100 basis points, so σ*= 21%.
  - Π_p^* =
  - Change in portfolio value = Π_p*–Π_p =
  -

22

Managing delta, gamma and vega

- Market makers – Buy and sell options to investing public. Market makers earn bid-ask spread on options written. Market makers do not want to assume risk, so they will want to hedge their portfolios. If investing public is net buy of options, market makers must be net sellers.
  – B-S partial diff equation: $\Theta + (r-q)S_0\Delta + ½S_0^2\Gamma = r\Pi$

- Q: How can market makers hedge (delta-neutral), if they tend to write options?
- A:
  – Recall that for delta-neutral portfolio: $\partial\Pi \approx \Theta\partial t + ½\Gamma\partial S^2$.
  – So, still exposed to large swings in asset price (since $\Gamma<0$).
  – would like to be gamma-delta neutral!

- Q: How can market maker acheive delta-gamma neutrality?
- A1:

- A2:
  –

23 # Hedging in Practice

- Hedging in Practice: "dynamic delta hedging"
  - Option market mkrs may ensure portfolios are delta-neutral at least daily.
  - Due to lack of depth, Γ and ν neutrality is difficult.
  - In practice, Γ and ν are monitored and improved if hit critical values.
  - Hedging is much less expensive (as %) for large portfolios.
  - Profits from B-S partial diff equation: $\Theta + (r - q)S_0\Delta + \frac{1}{2}\sigma^2 S_0^2\Gamma = r\Pi$

- Scenario Analysis - testing effect on portfolio value of assumptions about prices and volatilities.
  - Portfolio values for various prices and volatilities, esp extreme values.
  - Stress testing important, since extreme values occur often under-predicted.
  -

- Synthetic Options and Portfolio Insurance
  - Synthetic options - trades in equity portfolio (and T-bills) to match desired Δ.
  - E.g., Portfolio mgr wants protective put, which would decrease portfolio Δ.
  - Puts are expensive; may not track portfolio; insufficient horizon and depth.
  - Can also decrease Δ_p by buying T-bills; short index futures.

24 # Hedging: Portfolio Insurance

- Q: Suppose a portfolio with b=1.0 is currently worth \$100M. Suppose you insure this portfolio at \$950M (not incl div) with put options. $SP500_0 = 1,000$; K=950; T=70 days; r=6%; δ=2%; σ=20%; $\Delta_{put} = -0.2504$; p_0=\$13.69 via BS.

- Q1: How many puts must be purchased? Recall portfolio insurance:
- A1: # Puts = $b * \$Portfolio/\$100S_0$ =
  –

- Portfolio insurance is expensive
  – Alternatives? Simulate protective put by matching delta of insured portfolio
  – Rebalance as necessary, especially as simulated option matures.

- Deltas on equity portfolio, puts and insured portfolio
  – $\Delta_{equities} = b$*Portfolio/Index =
  – $\Delta_{puts} = \Delta_{put} * N_{puts}$ =
  – $\Delta_{portfolio}$ =

25 # Hedging: Portfolio Insurance

- Q2: How (else) can we reduce delta of portfolio to 75,000?

- A2: .
  - $\Delta_{equities}$* = b*Portfolio/Index + $\Delta_{TBillst}$

- Q3: How should we rebalance if share prices rise? If stock prices fall?

- A3: .
  –

- A3:

- Index Futures – maybe cheaper to transact than portfolio of portfolio.
  –

26 # Hedging: Dangers of Delta-Hedging I

- Portfolio Insurance and Market crashes
  - Synthetic puts popular in Oct 1987.
  - As shares fell, mgrs with synthetic puts sold shares to maintain target delta.
  - This effect was large enough to reinforce falling prices.

- Oct Crash Analysis
  - Wed (10/14/87) through Fri (10/16/87) market dropped 10%
  - Estimated $60B-$90B of equity assets managed with synthetic insurance.
  - This should have generated ~$12B sales in equity or index futures.
  - But only $4B had been sold.

- On Mon (10/19/87) – record volume on NYSE.
  - Sell programs from 3 portfolio insurers generate 10% of sales on NYSE.
  - Portfolio insurance generates over 20% of sales of index futures.
  - Insurers couldn't rebalance fast enough; put further downward pressure.
  - Mgrs owning outright puts were OK.

27

Hedging: Dangers of Delta-Hedging II

- Q: Suppose a bank sold European call on 100,000 shares of a stock, then it sold Euro put options on 100,000 shares. The share price is currently $48.67.
  S_0=48.67; K=50; r=5%; σ=20%; T=0.3836 yr; c=2.22751; p=2.60775;
  Δ_{call}=0.5000; Δ_{put}=–0.5000; Γ_{call}=0.6612; Γ_{put}=0.6612.
  – Calculate and interpret delta and gamma of this portfolio.

- A1: Calculate delta of portfolio
  Δ_p=

- A2: Calculate gamma of portfolio.
  Portfolio: Γ_p=

- Q: Suppose price rises to $51 (+2.3%). What happens?

- A: Π_p*–Π_p =

28 Hedging: Dangers of Delta-Hedging III

- Q: Suppose a bank sold for \$300,000 European call options on 100,000 shares of a stock with K=49. The bank then sells additional 100,000 European calls with K=51 and buys 200,000 calls with K=50. The calls expire in 1 day. The position earns an initial net cash flow of +\$59,000. The deltas are Δ_1=0.97430; Δ_2 =–0.5073; Δ_2=0.03051.
  - Analyze risk profile of position.
- A1: Calculate portfolio delta - Portfolio: Δ_p=
  –
- A2:
-
- Q: Will stress testing a work?
- A: