1 Black Scholes Option Pricing

- Read Hull chapters 12 and 13.
- Assumptions
- Model and Derivation
- Estimating Volatility
- Implied Volatility
- Dividends (see section 13.1 and 13.2)
- Currency Options (see section 13.5)

- Black-Scholes Formula

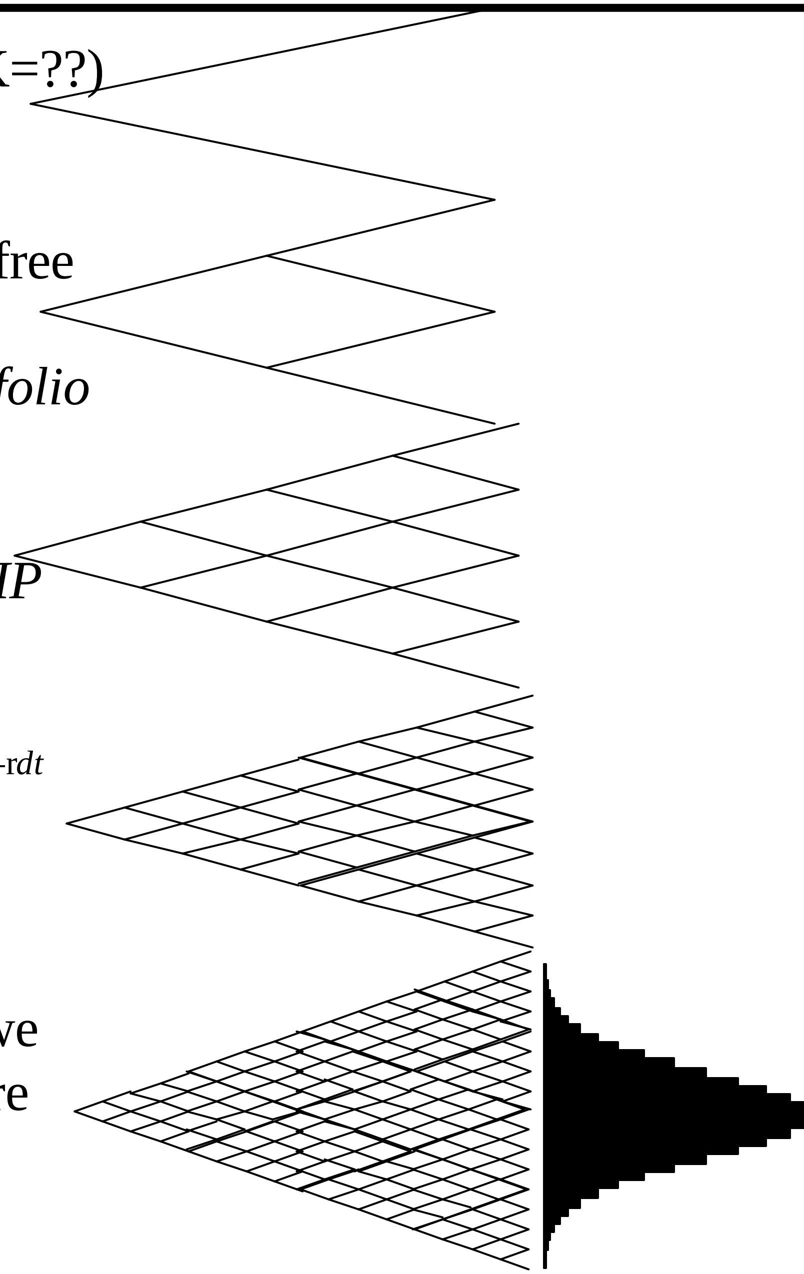
$$c_0 = S_o e^{-qT} \mathrm{N}(d_1) - \mathrm{K}\, e^{-rT} \mathrm{N}(d_2)$$

$$d_1 = [\ln(S_0/K) + (r-q+s^2/2)\mathrm{T}] / (s\ \mathrm{T}^{1/2}); \quad d_2 = d_1 - (s\ \mathrm{T}^{1/2})$$

- Put-Call parity
  - Put-Call parity with no dividends $c_0 - p_0 = S_0 - Ke^{-rfT}$
  - Put-Call parity with known dollar div $c_0 - p_0 = S_0 - Ke^{-rT} - \mathrm{PV}(Div)$
  - Put-Call parity with constant div yield $c_0 - p_0 = S_0 e^{-qT} - Ke^{-rT}$

2 Binomial Model Review

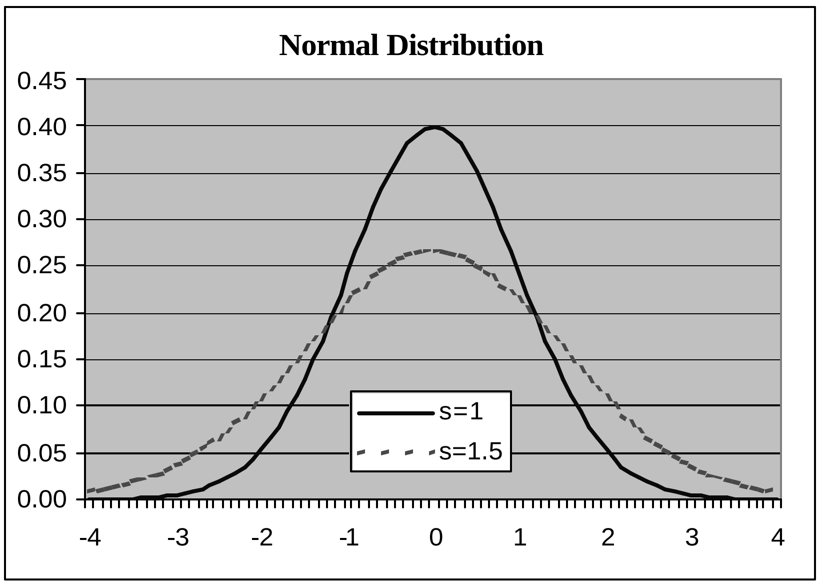
- Step 1: Consider a stock (S_0=??) and call (K=??)
  - compute payoff for call at each node.
- Step 2: Construct portfolio *replicating* risk-free investment.
  - *Hedge ratio* "Δ" for *riskless hedge portfolio*
    $\Delta = (f_u - f_d) / (S_u - S_d)$.
- Step 3: Price call by setting cost (PV) of *RHP* equal to discounted payoff.
  - Find fair price of call C_t at each node:
    PV of *riskless hedge portfolio* = FVe $^{-rdt}$
    $\Delta S_0 - f \quad = (\Delta S_0 u - f_u) * e^{-r(d\,t)}$
    $\qquad\qquad OR\ (\Delta S_0 d - f_d) * e^{-r(d\,t)}$
- If we choose up/down moves carefully, as we decrease time between nodes, possible future stock prices are distributed log-normal! (+/– tree moves not absolute!)

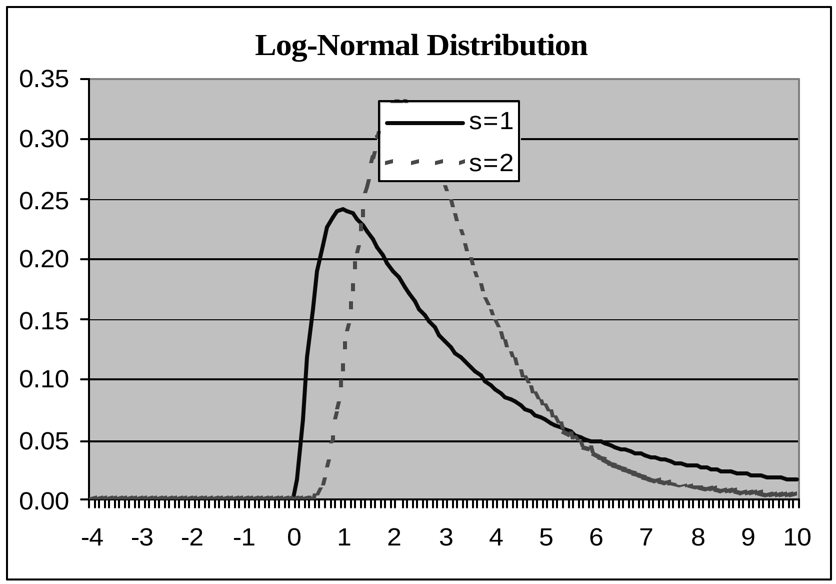


3 Assumptions for Black-Scholes

- Black-Scholes (B-S) model – option pricing formula derived under no-arbitrage conditions with set of formal assumptions, e.g., no taxes, transaction costs, margin:
  - Underlying is risky; no div; trades continuously in fractional units.
  - Risk-free rate, r, constant for all maturities. Volatility (σ) constant over time.
  - Option is euro; exercise price is K; expiration is T; Δ denotes small change.

- Distributional assumptions
  - C.c. return normally distributed $R_{0,t} = \ln(S_t/S_0) \sim N([\mu - ½\sigma^2]\Delta t\ ,\ \sigma\sqrt{(\Delta t)})$.
  - Then S_t is distributed log-normal, since $S_t = S_0 \exp(R_{0,t})$.
  - Note $\mu\Delta t$ is analogous to arithmetic avg, while $[\mu - ½\sigma^2]\Delta t$ is geometric avg.





4 # Assumptions – Are they Reasonable?

- Actual Stock Returns – histogram of index daily returns, with normal overlay.
  - Normality means 95% of daily returns between 0.04%–2% and 0.04%+2%
  - Stock returns tend to have big negatives, higher peak and fatter tails.
  - Technically, neg skew (–0.3), excess kurtosis (4.1); and heteroskedastic.
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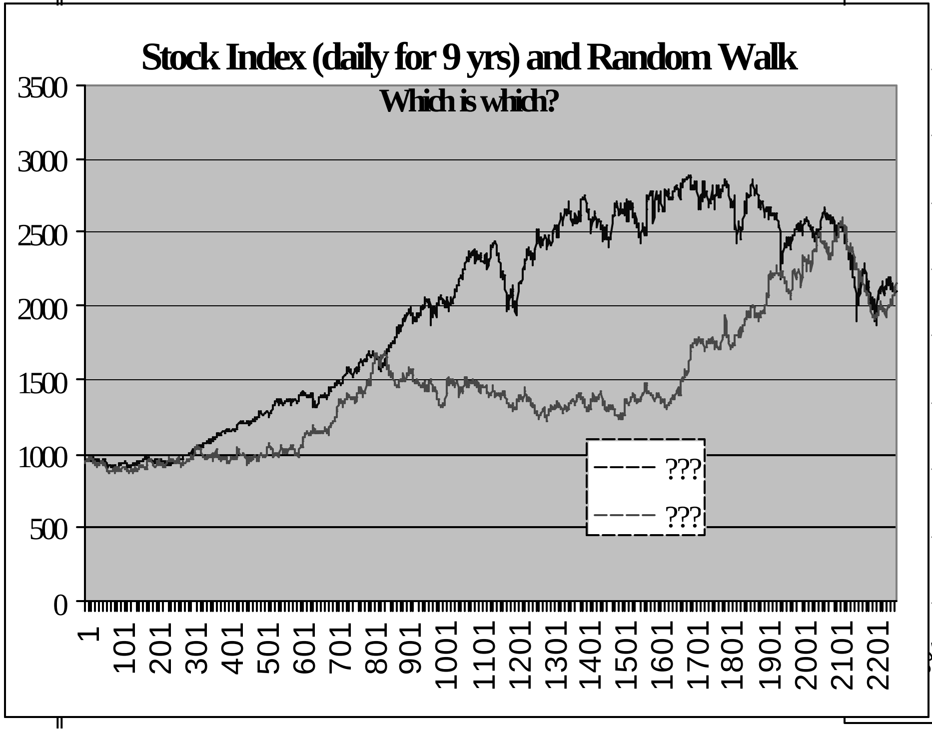

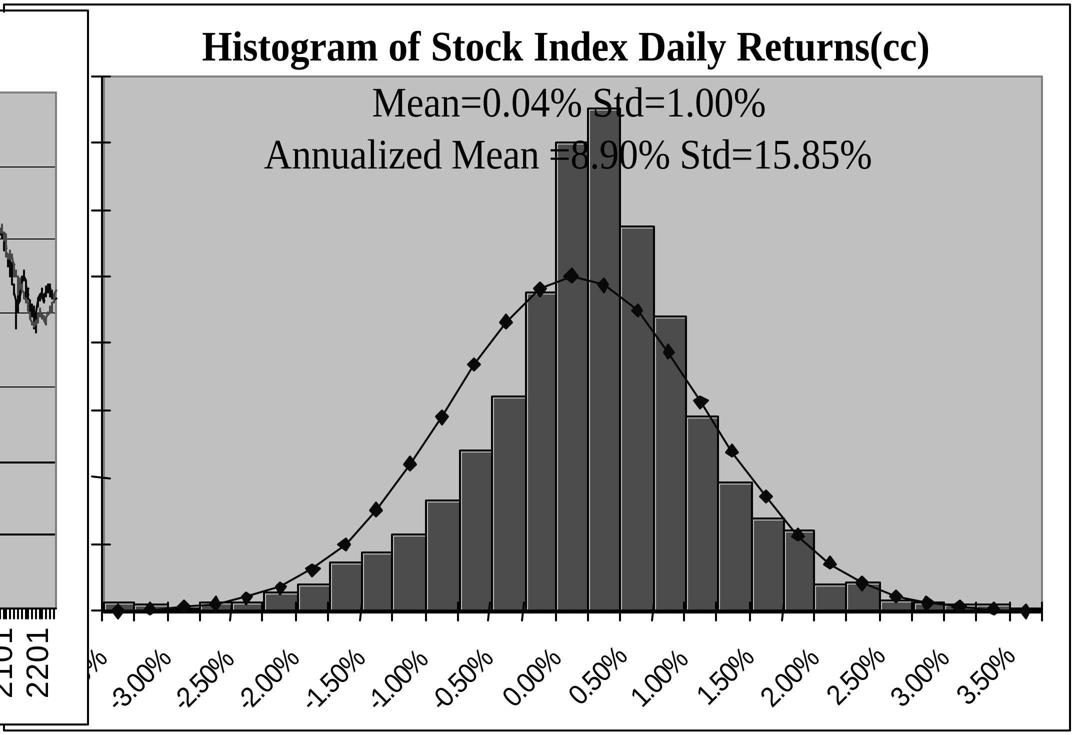

5 Black-Scholes Derivation

- **Derivation -** The objective is to derive an equation that expresses value of call option as function of its strike price and other variables.

- **Step 1.** Set-up riskless hedge portfolio long Δ shares and short one call option.
  - Denoting Π as current market value of portfolio, $\Pi = \Delta S - C$.
  - Change in value of portfolio is w.r.t to change in *S* is $d\Pi = \Delta dS - dC$.
  - To ensure riskless hedge portfolio, we find value of Δ such that $d\Pi = 0$.
  - $\Delta = dC/dS$.

- **Step 2.** Construct portfolio that *replicates* a risk-free investment.
  -

- **Step 3:** Price call option by setting cost (PV) of *RHP* equal to discounted payoff.
  - Price of call must satisfy second order partial differential equation.
  - Seldom easy to solve, but B-S-M redefined variables to give familiar form.
  -

6 Black-Scholes for Euro Options (with dividends)

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$p = -S_0 e^{-qT} N(-d_1) + K e^{-rT} N(-d_2)$$

$$d_1 = \frac{\ln\left(S_0 / K\right) + \left(r - q + \frac{s^2}{2}\right) T}{s\sqrt{T}} \qquad d_2 = d_1 - s\sqrt{T}$$

where

c_0 = current value of European call.

p_0 = current value of European put.

S_0 = current stock price

N(d) = probability that random draw from normal distribn is less than d.

K = Exercise price.

q = annualized div yield of underlying stock (text ignores div yld for now).

e = 2.71828, base of natural log.

r = Risk-free interest rate matching maturity of option, c.c.

T = time to maturity of option in years.

σ= Std deviation of return on stock, c.c.

Note: Black-Scholes is derived for European calls! No value for early exercise!

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Black-Scholes Characteristics 1

- What is Black-Scholes? $c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$
  - $N(d_i)$ interpreted as risk-adjusted probability option will expire in-the-money
  - Recall that value of call is discounted expected payoff, $Max(0, S_T - K)$.
  - Suppose for below that $q=0$ (no dividends).

- Suppose there is low probability call option will be exercised ($S_0 << K$).
  -

- Suppose there is high probability call option will be exercised ($S_0 >> K$).
  - Then $N(d_1)$ and $N(d_2)$ are close to one, and value of call is $c_0 = S_0 - Ke^{-rT}$
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- Using Black-Scholes – derived for European options
  - American calls on non-div stocks not exercised prior to expiration.
  -

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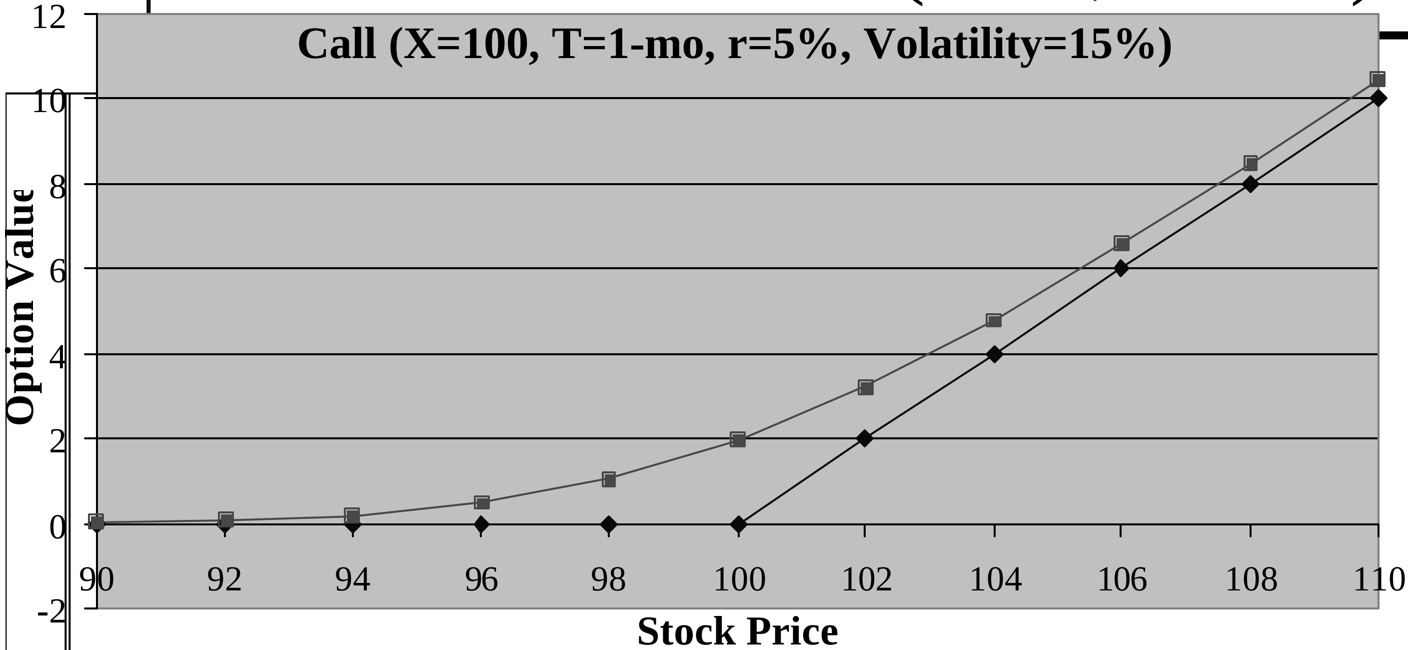
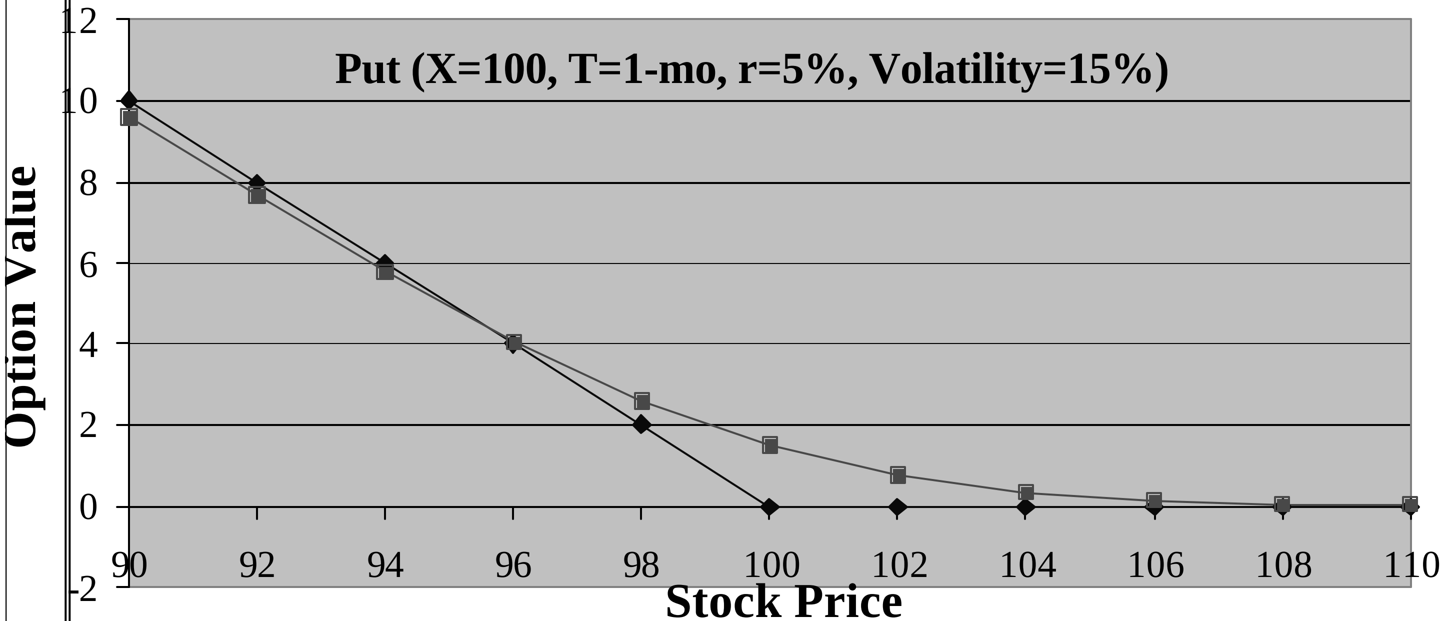
Black-Scholes Prices (Euro, no div)





Parity for Euro:
$c - p = S_0 - K e^{-rT}$

Black-Scholes:
$c = S_0 N(d_1) - K e^{-rT} N(d_2)$

$N(d_i)$: risk-adj prob option expires in-the-money

Euro Put: possible early exercise exceeds value if deep in-the-money.

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Black-Scholes Example

- Suppose S_0 = \$50, K= \$45, T = 6 months, r = 10% c.c., and σ=28%. Calculate the value of a call and a put option.

- Black-Scholes $c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$

  $d_1 = [\ln(S_0/K) + (r-q+s^2/2)T] / (s\ T^{1/2})$; $d_2 = d_1 - (s\ T^{1/2})$

$$d_1 = \frac{\ln\left(50/45\right) + \left(0.10 - 0 + \frac{0.28^2}{2}\right)0.50}{0.28\sqrt{0.50}} = 0.884$$

$$d_2 = 0.884 - 0.28\sqrt{0.50} = 0.686$$

$c_0 = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$

$c_0 =$

$=$

$=$

$p_0 =$

$=$

Note: Excel worksheet fn =NORMSDIST(x), but need to load Analysis Toolpack.

10 # Black-Scholes – Calculating interest rate

- Compute interest rate from T-bills, based on avg of bid/ask

Maturity	**Days Maturity**	**Bid**	**Ask**

- Compute average of bid and ask
  -
- Compute price of 100 par value bill based on bank discount rate.
- Compute the EAR
  - EAR =
- Compute the continuously compounded, annual rate of return.
  - (1+EAR) =
  -
-

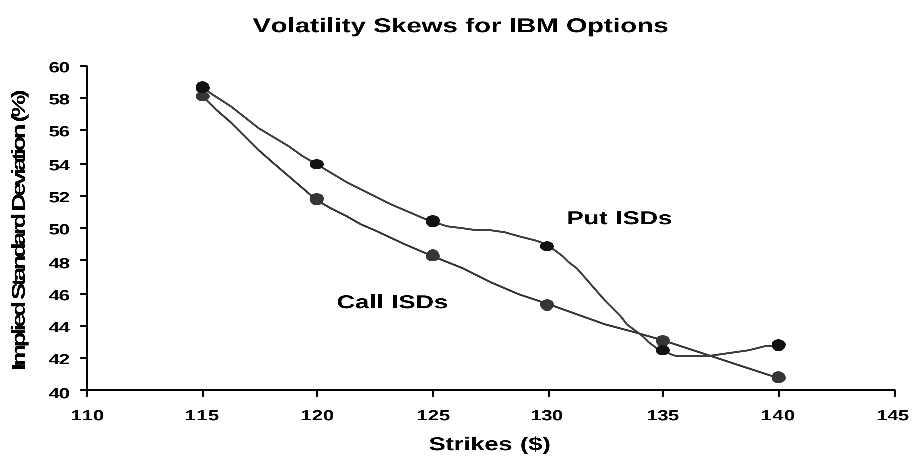
11 # Black-Scholes – Estimating Volatility

- $c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$; $d_1 = [\ln(S_0/K) + (r - q + s^2/2)T]/(s\,T^{1/2})$; $d_2 = d_1 - s\,T^{1/2}$

- Estimating volatility - Let S_t be daily price and let $u_t = \ln(S_t/S_{t-1})$ be daily return.
  - Use 90-180 trading days, or maturity. Discard data around ex-div date (tax).
  - Find daily variance of u_t; multiply by 252 days; take sqrt(s^2) for std dev.

$$s^2_{annual} = 252 * s^2_{daily} = 252 * \frac{1}{N-1} \sum_{i=1}^{N} \left(u_t - E(u_t)\right)^2$$

- Implied Volatility – level of volatility implied by B-S or binomial model.
  - May avg from several liquid options on same asset; used to price less liquid.
  - Issues: non-simultaneity; bid-ask; model mispecification.
  - VIX –
  - Volatility skew -



12 # Black-Scholes with Dividends

- Black-Scholes
  - $c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$
  - $d_1 = [\ln(S_0/K) + (r - q + s^2/2)T] / (s\ T^{1/2})$; $d_2 = d_1 - (s\ T^{1/2})$

- Dividend Yield – basic Black-Scholes formula does not include dividends.
  - Accounting for div yield – discount current stock price by div yield.
  -

- Dividends paid as known dollar values –
  - Recall that stock price (and call value) drops on ex-div date.
  -
  -

- Black adjustment for dividends – devised by Fischer Black before PC era.
  - Price option as greater of
    (1) Euro option
    (2) Euro maturing just prior to latest ex-div date.

13 # Binomial Model vs Black Scholes

- Binomial Model vs. Black Scholes
  - Euro call no div –
  - American calls on div paying stocks –
  - Euro and American puts –

- In practice – computing power is now cheap.
  - Analysts use binomial model to price American and other options.
  - Consider American call/put with K=100; S_0=100; T=6-mo; r=5%; σ=25%.

	Call Prices		Put Prices	
Time Steps	Binomial	Black-Scholes	Binomial	Black Scholes
5	8.601	8.26	6.334	5.791
10	8.087	8.26	5.932	5.791
15	8.373	8.26	6.128	5.791
20	8.173	8.26	5.977	5.791
30	8.202	8.26	5.992	5.791
50	8.225	8.26	6.004	5.791
99	**8.277**	**8.26**	**6.039**	**5.791**

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Tips on Pricing Options for the Savvy Investor

- Black-Scholes is powerful formula for pricing Euro options
  - Derived before binomial model.
  - Some assumptions are strong (continuous trading, normally distributed cc returns), but model works well.
  - No arbitrage assumption cornerstone in finance and revolutionized field.
  - B-S can be viewed as special binomial model with many steps.
  - B-S less useful for pricing Amer puts and calls on div paying stocks.

- There are many adjustments to Black-Scholes, but still very useful benchmark.
  - Options on stock indexes used with dividend yield
  - Options on indiv stocks often adjusted by subtracting PV of div for S_0.

- Black-Scholes Formula: $c_0 = S_o\, e^{-qT} \mathrm{N}(d_1) - \mathrm{K}\, e^{-rT} \mathrm{N}(d_2)$

  $d_1 = [\ln(S_0/K) + (r-q+s^2/2)\mathrm{T}] / (s\ \mathrm{T}^{1/2})$; $d_2 = d_1 - (s\ \mathrm{T}^{1/2})$
- Put-Call parity
  - Put-Call parity with known dollar div $c_0 - p_0 = S_0 - Ke^{-rT} - \mathrm{PV}(Div)$
  - Put-Call parity with constant div yield $c_0 - p_0 = S_0 e^{-qT} - Ke^{-rT}$

15 Standard Normal Probabilities

TABLE 5.1 Standard Normal Probabilities

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.0851	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8860	.8888	.8907	.8925	.8943	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990