Determination of Forward and Futures Prices

- Introduction
  - Investment assets vs consumption assets.

- Pricing forwards by arbitrage
  - Cash and carry (long asset, short futures) has no risk.
    - Return should equal risk-free rate (T-bill or LIBOR).
  - Application to stock index arbitrage.

- Issues
  - Forwards vs futures.
  - Cost of carry and convenience yield
  - Futures price and the future spot price

- Notes - Quotes at http://www.futuresource.com and http://www.cme.com

- Notation
  - S_0: Spot price today F_0: Futures or forward price today ($F_{0,T}$)
  - T: Time until delivery date r: Risk-free interest rate for maturity T

Arbitrage

- Review of "selling short" - involves selling securities you do not own.
  - Your broker borrows securities from another client and sells on open mkt.
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- Pure Arbitrage – earn profits with no net investment and no risk.
  - Investments *with same risk (or no risk)* must have same expected return.
  - Otherwise, go long cheap asset and short expensive asset, earn arb profits.
  - Suppose T-bill (borrow-lending) rate is 5% and an investment returns 10%.....
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- Arbitrage arguments accurately price futures on assets on investment assets.
  - Investment assets - held by significant numbers for investment purposes.
    - Ex: stocks, bonds, gold, silver.
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- Assumptions – need only to hold for few major mkt participants.
  - no transactions costs; same tax rate on all profits.
  - borrowing and lending rates are the same risk-free rate.
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Forward Prices by Arbitrage 1

- Arbitrage? - Suppose 12-month risk-free rate is 4.9% c.c. (T-bills or Eurodollars).
  - Spot price of stock is $100 and quoted 12-month forward price is $110.

- Consider two risk-less investment strategies for over 12-months.
  - **Strategy (1)** Long T-bills: long $100 T-bills. Earn $100*e^{0.049x12/12}$ = $105.
  - **Strategy (2)** Long asset, short forward: long $100 stock. Lock-in sale of $110 This is known as "cash and carry" (buy asset and sell forward).
- A:

		CF(0)	CF(T)	Gross Return
Short (1)	Borrow S_0 at risk-free rate			
Long (2)	Long asset plus short futures			
		0		Returns not equal!

- Arbitrage? - Suppose now the quoted 12-month futures price is $104.
- A:

		CF(0)	CF(T)	Gross Return
Long (1)	Invest S_0 at risk-free rate			
Short (2)	Short asset plus long futures			
		0		Returns not equal!

4 Forward Prices by Arbitrage 2

- Arbitrage? - Suppose 12-month risk-free rate is 4.9% c.c. (T-bills or Eurodollars).
  - Spot price of stock is \$100 and quoted 12-month forward price is \$105.

- Consider two risk-free investment strategies for over 12-months.
  - **Strategy (1)** Long T-bills: long \$100 T-bills. Earn $100 * e^{0.049 \times 12/12}$ = \$105.
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- A: .

		CF(0)	CF(T)	Gross Return
Short (1)	Borrow S_0 at risk-free rate			
Long (2)	Long asset; short futures			
		0		Equal!

Symbols		**CF(0)**	**CF(T)**	**Gross Return**
Short (1)	Borrow S_0 at risk-free rate	$+S_0$	$-S_0 e^{rT}$	e^{rT}
Long (2)	Long asset; short futures	$-S_0$	$+F_{0,T}$	$F_{0,T}/S_0$
	Note: $F_{0,T} > S_0$	0	$F_{0,T}-S_0 e^{rT}=0$	$\mathbf{F_{0,T}/S_0 = e^{rT}}$

Forward Prices by Arbitrage 3

- Notation
  - S_0: Spot price today; F_0: Futures or forward price today ($F_{0,T}$)
  - T: Time until delivery date; r: Risk-free interest rate for maturity T (c.c.)

- For any investment asset that with no income no storage costs:
  - No arbitrage implies $F_0 > S_0$. Compare buying spot vs long forward (invest $).
  - Return on (2) [long asset, short future] = Return on (1) [T-bills]

$$F_0 = S_0 e^{rT}$$

- Forward contract on invest. asset earning known dollar income
  Let I be PV of income in dollars

$$F_0 = (S_0 - I)e^{rT}$$

- Forward contract on invest. asset earning known yield and no storage costs.
  Let q be avg unit yield over contract. Long asset grows by e^{qT} so short $F_0 e^{qT}$ units.

$$F_0 = S_0\, e^{(r-q)T}$$

Forward Prices by Arbitrage 4

- S_0: Spot price today; F_0: Futures or forward price today ($F_{0,T}$)
  T: Time until delivery; r: Risk-free rate for maturity T (c.c.)

- For any investment asset that with no income no storage costs:
  - Return on long asset, short future [F_0/S_0] = Return on T-bills [e^{rT}]
    $F_0 = S_0 e^{rT}$
- Forward contract on invest. asset with income earning known dollar income
  $F_0 = (S_0 - I)e^{rT}$ where I is the PV of income
- Forward contract on invest. asset earning known yield and no storage costs
  $F_0 = S_0\, e^{(r-q)T}$ where q is avg unit yield during contract (c.c.)
  long asset grows by e^{qT} units; short $F_0 e^{qT}$ units to lock-in price for all units.

- Forward on Commodities with storage costs, i.e., negative income or yield.
  - $F_0 = S_0\, e^{(r+u)T}$ where u is storage cost per unit time as % of asset value.
  - $F_0 = (S_0 + U)e^{rT}$ where U is PV storage costs; e.g., $1/oz/yr payable in arrears.
  - Gold and corn can be stored, but some commodities cannot (electricity).
- Futures and Forwards on Currencies – Price of FX in $US -- buying FX.
  - $F_0 = S_0 e^{(r-rf)T}$ where r_f is foreign risk-free rate.
  - Intuition: currency appreciation must match interest rate differential.

Forward Prices by Arbitrage 5

- Suppose 3-month annualized risk-free interest rate is 5% c.c. (T-bills), and 3-month single-stock forward contract trades on MSFT.

- Q: Suppose MSFT has a spot price of $40. What is the forward price?
- A: F_0 =

- Q: Suppose MSFT pays a dividend, today, of $1. What is the forward price?
- A: F_0 =

- Q: Suppose MSFT pays dividend in 3-months of $1. What is the forward price?
- A: F_0 =

- Q: Suppose MSFT pays a dividend, expressed as an annualized yield of 10%, c.c. which is reinvested (e.g, DRIP). What is the forward price?
- A: F_0 =

- Note: Because div yld is greater than risk free rate, forward price is below spot.
  - Compare buying asset (earning div yld) vs long forward (earning risk-free).

Forward vs Futures Prices

- Futures contract – no arbitrage equations actually price newly marked forward.
  - Futures with mark-to-market is analogous to issuing forward contracts, closing at day-end no arb value, and reissuing with updated delivery price.
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- Strong positive (neg) corr between interest rate and asset price S_0 implies futures price is slightly higher (lower) than forward price. Evidence mixed.
  - Ex: Suppose S_0 increases, increasing F_0. Futures profit realized immediately.
  - Positive correlation implies profit reinvested at higher interest rate.

- Recall CME SP500 mini futures, with r=10%, q=1%, maturing 1/4/02.

Time to Maturity (yrs)	Date	SP500 Close	Futures / Forward Price	CF on Long Future	Cumm Gain	Margin Act Balance	CF on Long Forward
0.008	1-Jan-02	1000	$1,000.74			$3,565.00	$0.00
0.005	2-Jan-02	1005	$1,005.50	$237.79	$237.79	$3,802.79	$0.00
0.003	3-Jan-02	1010	$1,010.25	$237.67	$475.45	$4,040.45	$0.00
0.000	4-Jan-02	1015	$1,015.00	$237.55	$713.00	$4,278.00	$713.00

Stock Index Futures

- Stock index futures
  - S&P 500 futures $250*index and $50*index (mini) on CME
  - Nasdaq 100 futures $100*index and $20*index (mini) on CME
  - CME futures trade virtually 24 hrs/day on Globex. NYSE opens 9:30am.

- Stock index – viewed as investment asset paying dividend yield.
  - Futures satisfy $F_0 = S_0\, e^{(r-q)T}$, where q is div yield on portfolio in index.
  - Div yield estimated over contract life, though most paid Feb, May, Aug, Nov.
  - Index arbitrage is popular strategy for well-capitalized institutions.
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- Q: What is fair value for SP500 futures with delivery in 5-mo (settles Jun 20), if on Jan 20 SP500 is 886.30; div yield is 1.77%; 5-mo LIBOR is 1.37%.
  A: $F_0 =$

- Q: Suppose, in early morning trading tomorrow, this futures contract trades at 890. What would you expect to the stocks in the SP500 to open at?
  A: 890 =

Stock Index Arbitrage

- **9:00AM:** S&P futures vs fair value: +0.9. Nasdaq futures vs fair value: +7.0. The futures market lifts off its lows and continues to point to a higher open for the cash market. The economic data had little effect on the market, which is holding its breath ahead of tomorrow's Employment report. The generally better than expected same store sales reports are supporting the favorable bias.

- **9:15AM:** S&P futures vs fair value: +0.7. Nasdaq futures vs fair value: +5.5. Having changed little through the morning, the futures market continues to point to a higher open for the cash market.

- The direction indicated by the futures market were correct. In the first few minutes of trading, SP500 open about 1 point higher. Nasdaq opened about 5 points higher.

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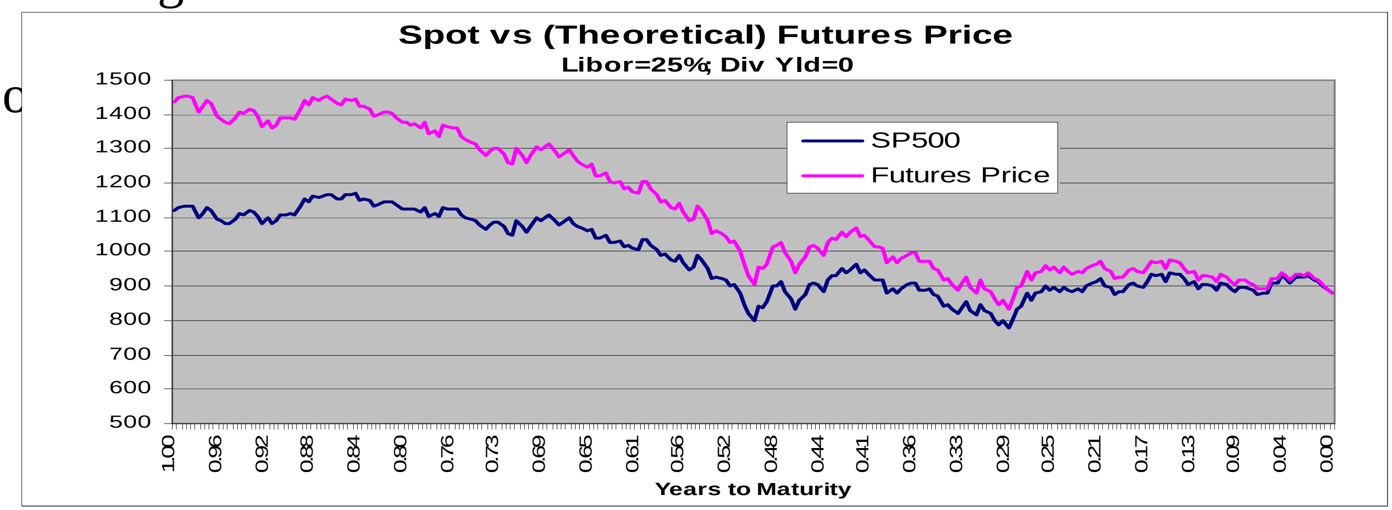

Futures Price on Bond

- Forward contract on invest. asset earning income $F_0 = (S_0 - I)e^{rT}$.

- Q: Suppose 6-mo, 12-mo annualized interest rates are 9%,10% c.c. The spot price on a bond (S_0) is $900, F_0 is $930 and the bond pays $40 coupon in 6- and 12- mo.
  - PV of coupon:
  - FV of coupon:

- A: Futures price should be $F_0 = (S_0 - I)e^{rT} =$
  - $F_0 > S_0$ makes sense… But quoted futures price is too high ($F_0 > (S_0 - I)e^{rT}$)
  - Strategy (1)
  - Strategy (2)

		CF(0)	CF(T)	Return
Short (1)	Borrow S_0 at risk-free	+900		
Long (2)	Long asset; short futures	–900		
		0		

Forward Contract Valuation

- Forward contracts - analysis derived delivery price consistent with no arb.
  - What if delivery price specified in contract, K, differs from this F_0?
  - If $K < F_0$, then long position would pay extra today (up to PV of $(F_0 - K)$).
  -

- Value of a long forward contract, f, is $f = (F_0 - K)e^{-rT}$
  Value of a short forward contract is $(K - F_0)e^{-rT}$
  - K is delivery price in a forward contract &
  - F_0 is forward price that would apply to the contract today

- Forward delivery price – contracts initially written with $K = F_0$.
  - So, short and long forward positions do not initially exchange cash.
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- Value of long forward contract, $f = (F_0 - K)e^{-rT}$ for investment asset with
  - no income no storage costs: $f = S_0 - Ke^{-rT}$
  - earning known dollar income; no storage costs: $f = S_0 - I - Ke^{-rT}$
  - earning known yield and no storage costs: $f = S_0e^{-qT} - Ke^{-rT}$

Futures on Consumption Commodities

- Futures on Consumption Commodities (not widely held as investments).
  - Firms may not sell inventory, if operations depend on physicals (e.g., oil).
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  - $F_0 \leq S_0 \, e^{(r+u)T}$ where u is storage cost per unit time as % of asset value.
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- Two concepts can simplify the notation and number of equations.
  - Cost of carry (c) – the storage cost plus interest costs less income earned.
    $c = r + u$; or $c = r - r_f$; $c = r - q$; $c = r + u - q$... as appropriate
    $c = r + u - q$ is relevant for gold, which can be "leased" at rate q.
  - Convenience yield (y) –
  - So $F_0 = (S_0 - I)e^{(c-y)T}$ or $F_0 = (S_0 + U)e^{(c-y)T}$

- Delivery Timing – futures specify deliver period, with short choosing date.
  - Will maturity date tend to be early or late?
  - If futures prices are increasing with maturity, then $(c-y) > 0$,
  - Short position prefers to delivery early (return on cash dominates).

Commodity Futures Prices

- Electricity – Futures for 1-day ahead delivery of 1 megawatt hr on June 2001. For 4am delivery, price $31.97. For 4pm delivery, price 169.52. Arbitrage?

- Natural gas – production constant, but demand heavy in winter.
  - Forward prices tend to rise in Fall, suggesting storage.
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- Ag commodities – relatively constant demand but seasonal variations in supply.
  - Forward prices increase prior to next harvest, then drop (as storage ceases).
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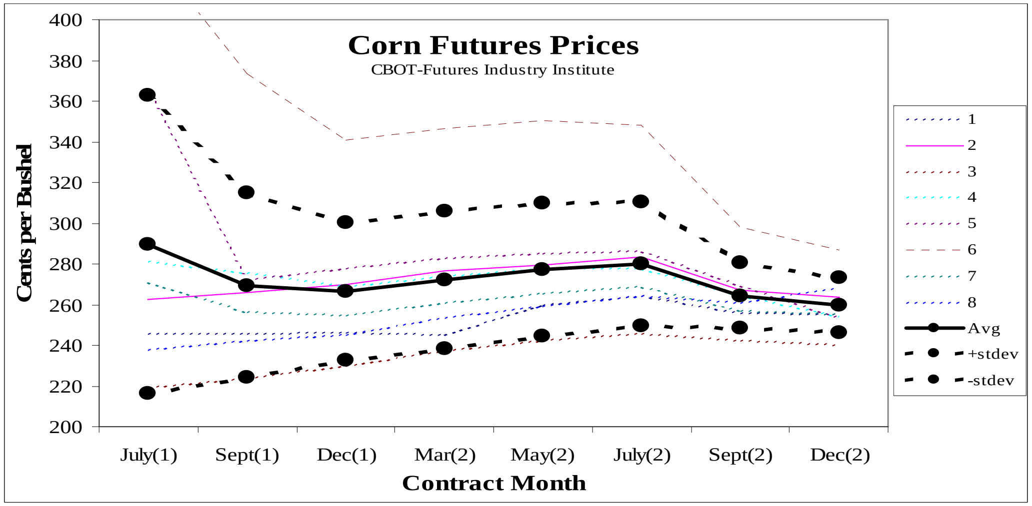

Futures Prices & Expected Spot Prices

- What is relationship between futures price (F_0) and expected spot $E(S_T)$?
  - Backwardation (inverted): $F_{0,T} < S_0$; Normal backwardation: $F_{0,T} < E(S_T)$
  - Contango: $F_{0,T} > S_0$; Normal Contango: $F_{0,T} > E(S_T)$

- Keynes – Speculators require compensation for risk. Hedgers accept neg return.
  - If speculators long, $F_0 < E(S_T)$; and futures price rises faster than risk-free rate.

- Portfolio Theory – required (equil) return depends on systematic risk.
  - If systematic risk is pos, return on long futures > risk-free rate and $F_0 < E(S_T)$
  - If no systematic risk F_0 is an unbiased estimate of S_T.
  - Empirical evidence -

- Note: commodity spreads – trading in inputs/outputs based on same commodity.
  - Crush spread – e.g., long soybean and short soybean oil or meal. “Trading the crush”.
  - Crack spread –

Summary

- S_0: Spot price today; F_0: Futures or forward price today ($F_{0,T}$)
  T: Time until delivery date; r: risk-free interest rate for maturity T

- Generalized futures pricing equations
  - Investment asset $F_0 = (S_0 - I)\, e^{cT}$ where I is PV of dollar income on asset
  - Consumption asset $F_0 = (S_0 - I)\, e^{(c-y)T}$ with y=0 for an investment asset.

- Special cases
  - For asset with known yield $c = r - q$ where q is avg unit yield during contract.
  - For foreign currency $c = r - r_f$ where r_f is foreign risk-free interest rate.
  - For commodities $c = r + u$ where u is storage cost/unit time as % of asset value.
  - For commodities $I = -U$ where U is PV of dollar storage costs.

- Value forward contract after issue ($K \neq F_0$)
  - Value long forward contract $f = (F_0 - K)e^{-rT}$
  - K is delivery price in a forward contract &
  - F_0 is forward price (no arb) that would apply to contract today