THE PRESERVATION OF TRUTH

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Abstract. On the classical account of validity, an argument is invalid iff there is a model in which its premisses are true but its conclusion false. A rigid adherence to this account has led some paraconsistentists, the so-called *dialetheists* to account for the invalidity of *ex falso quodlibet* (efq) by recourse to models that satisfy classically inconsistent sets of sentences on non-classical interpretations of negation or conjunction or both. Preservationists have generally countered that the problems with *ex falso quodlibet* are application-specific and would be better solved by requiring an inference relation to preserve mathematically well-defined semantic measures of sentence-ensembles.

It is undeniable that formal manipulations such as those of the dialetheists will yield efq-free sublogics of classical logic. It is however deniable that their model theory confirms any metaphysical claim about the tolerances of truth or the meaning of negation. In fact it is arguable that folk-truth need play no very serious role in the formal sciences where designated values will do, or in the physical sciences where confirmation is all that can be had. No one has yet given a theory of truth capable of playing an active rather than token role in formal semantics.

This paper reintroduces an essentially Leibnizian conception of truth as the satisfaction of countably infinite sets of standards, and explores the consequences for truth-based semantics of inference of the standards of truth that are adopted. The account is realized in models that assign to atoms valuations registering, for each standard, whether the atom satisfies it or does not. It is shown that semantically licensed proof-theoretic behavior of propositional connectives depends upon the properties of adopted standards of truth, even when (a) compositionally classical truth-conditions for connectives are retained, and (b) inference is required only to preserve truth.

1. Truth in animadvertising

'What is truth? said jesting Pilate, and would not stay for an answer', wrote Francis Bacon, and did not go on to give one. His implicit assumption, that only a jokester could claim to be at a loss about truth must be a fundamental of theology. Certainly the idea of truth is a hot property for the religionist and a source of warmth for the religious. But in religion this is only what we should expect, for no greater detail can be hoped for in matters of religious doctrines. Since we cannot expect to learn of them, say, what they mean, or what their detailed physical significance is, or how in detail they come about, there can be little more to be said of specifically theological claims than that they are true. At any rate, for the purposes of religion, an assurance that they are true seems to be sufficient.

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But even among the secular humourists of philosophy there is a kind of confidence that there is nothing much worth knowing about truth beyond our intuitive understanding. Questions such as 'Are any value-judgements true?' Not that there has been no progress on these scores. Reflective religious adherents realize that they must take on faith that their favourite doctrines mean anything at all. And secular intellectuals have reluctantly accepted that they don't really know what their talk of truth is about. If Bacon thought Pilate must be joking because he thought the answer was obvious, Mill at least characterizes Pilate's question as 'scornful', on the grounds that the inquiry into truth is 'difficult and noble'([10], 98) So, as compared with Bacon's response, Mill's is the more sympathetic to our own doubts on the matter. Whatever one might be inclined to say on the point of nobility, that the nature of truth is difficult can safely be placed beyond dispute.

But we go further than Mill in our exoneration of Pilate. It is merely tiresome of anyone to speak of having come into the world to bear witness to the truth, or to claim to be heard by people who are of the truth. Especially *in gallicantu*, but surely at any time of the day, such talk would set anyone's teeth on edge who had given the matter any thought. So Pilate represents, to the theological outsider at any rate, not so much an icy douche as a refreshing eddy. Truth is no more our bath water than it was Pilate's, and, with all due respect to Jeez and Bacon and all who, like Rupert Brooke, find benison in hot water, we do not know whose bath water it has been.

Now of course at a conversational level we know how the language of *truth* is to be used. So, to disarm the obvious rejoinder we do want to say clearly here no more than what we take to be true, and we want to be neither dishonest nor mistaken. But this can only be to say that we are engaged in the usual way in trying to say something, and we want what we say to satisfy the usual standards for such attempts. That much we can say without a studied theoretical understanding of what it is for something to be true, or of what those propositional somethings are to which truth is to be attributed. We have only what our early upbringing has given us: some command of the language, and some standards of intellectual deportment. It would be delicious to have more, but philosophy has given us no instruction. There were the usual desultory lectures about coherence and correspondence, but neither of these was defined, and we quickly passed on to other topics, those two analyses having been found unsatisfactory and interest having been exhausted.

We need not dwell on the fact that for the physical sciences, truth plays no very serious role. Suffice it to say that although like us, scientists have never had a studied account of what truth is, science has rubbed along, even had the odd success, without one. Certainly scientists want themselves to be understood, particularly by ultimate sources of research funds, as searching for the truth. Who does not? After all, it's true. But if truth plays any more substantial role than that, it is evidently a role for which no deep understanding of its nature is required.

2. Truth in Logic

Now the language of *truth* has always had an honourable place in the culture of logical theorizing. The pursuit of logical truths was traditionally its chief end. But as a part of the subject matter of logic, it was always doomed to give way to the hierarchical language of satisfaction and validity. And in the characterization of the status of metatheorems, or of deep assumptions such as that of the axiom of choice, it does no logically essential work. The label *logically true* has spent its later career seconded to general philosophy as a label for banalities, from which work it seems mercifully to be passing into early retirement.

Again, the notion that a system of inference is sound if its provability relation preserves truth, accounts historically for much of the philosophical interest in such systems, even for philosophers for whom, as for Mill, truth remains nobly problematic. And perhaps for applications where truth has useful conversational traction, that traditional notion of soundness should persist as a desideratum of strategic inferential planning. But the success of a system of logic in such applications does not depend upon a philosophically recognizable doctrine of truth. Its inference relation need only satisfy the condition that for any input for which every function on the left of the turnstile outputs a 1, some function on the right of the turnstile should also output a 1. *Truth-preservation* provides a *reading* of that requirement, but it is only a reading and not an interpretation. It cannot be an interpretation without some independent understanding of what truth is.

To be sure, even logicians distrustful of *truth* or disdainful of metaphysics might yet refer to the set *2* as the set of *truth-values*. After all, the semantic role of the model depends upon its mathematical properties, not upon our conversation about them. But there are also logicians for whom the language of truth is not merely convenient habit or noble end, but a source of uberous conceptual nourishment. And, it hardly needs saying, dreamily steeping in the bath water, one is naturally susceptible to the insidious charms of the rubber ducks, particularly that of truth-preservation. This then is the difference between the bath party and the rest: for the former the goals of logical theory are defined by their philosophically charged reading of the designated value *1*; for the rest, the reading in itself holds no irresistible charm, while the intellectual history of the mathematics is insufficient to dissuade them from wanting to explore alternatives to it.

3. The Preservationist Intervention

That, wisecracks aside, is a pretty fair account of the difference between what have been called the *aletheist* and the *preservationist* approaches to logical theory. The working hypothesis of the former is that the only legitimate semantic aim of any system of logic is that its $\vdash$ (alternatively its $\rightarrow$) should preserve truth; that of the latter that systems of logic should be designed whose $\vdash$'s and $\rightarrow$'s preserve whatever suitable, mathematically well-defined properties interest us in the proposed domains of application. What makes a property suitable must not be circumscribed too closely, but one obvious requirement is that the property, like classical satisfiability, must be *naturally non-monotonic*, that is non-monotonic along set-inclusion; for a property that was *preserved* under the operation of forming supersets would not permit the distinction between inferrability and non-inferrability. Now it is evident that the differences between the two approaches must emerge most dramatically in applications that from time to time encounter inconsistent sets of premisses, since, on the face of it, the preservation of truth can place no constraint at all upon inference from premisses that cannot be true, nor could the preservation of *any* property that, like truth, was incompatible with inconsistency. Since there are suitable properties that are not incompatible with inconsistency, the preservationist's intuition that inference from inconsistent ensembles can be constrained is at least mathematically well-founded. By contrast, for the aletheist, the corresponding intuition must inevitably require what classicists would regard as a radical model-theoretic adjustment, one that will permit all of an inconsistent ensemble of sentences to be simultaneously satisfied. Since the aletheist is philosophically committed to an alethic reading of satisfaction, their notion of truth must acquiesce in the adjustment. If only truth-preservation constrains inference, then if our intuitions about inferrability tell us that inference from inconsistent ensembles is constrained, then those intuitions about constrainedness must be made to inform our intuitions about the extension of truth, certainly our intuitions about the conditions under which the sentences of the formal language are satisfied, particularly conjunctions and negations. In short, the account of the conditions of what makes a negation or a conjunction true must be modified in such a way as to permit $\alpha \wedge \neg\alpha$ to be true if we are to reject the inference from $\alpha \wedge \neg\alpha$ to arbitrary β.

Now even if we have no very well worked out theoretical account of truth, we can nevertheless have a pretty clear idea of the truth-conditions of *and* and *not.* The reason is that our account of those truth-conditions is grounded in our ordinary conversational understanding of truth. Thus, for example, for the ordinary run of sentences, and in our ordinary way of speaking, a sentence is true if and only if its negation is not. The aletheist must, therefore, send out for some new intuitions based on counterexamples to the old. We do not here wish to argue that none will be forthcoming, only that we not waste the meantime by ignoring alternative avenues of research. The suggestion that some special as yet unformulated sentences of a known physical theory, or those of some abstruse and as yet unformulated physical theory, might not obey that principle is hardly an argument for the elaboration, just on spec., of a highly particular class of models based upon an equally highly particular amendment of our ordinary conceptions. Whence, it might be asked, come the highly particular intuitions as to what should and what should not be theorems? Shall they be decided upon, like the doctrines of the merged churches of St. Asaph and St. Osaph, by a majority vote of the common shareholders, or perhaps by a survey of the ASL?

The model-theoretic recipe (schematically)

$$(\emptyset) \models^{\mathfrak{M}}_{x} \alpha \wedge \beta(\neg\alpha) \Leftrightarrow \gamma$$

presents only two points of adjustment. We can adjust γ or we can adjust $\models^{\mathfrak{M}}_{x}$. In a manner of speaking, both sorts of adjustment have been studied. Some have added structural features to the universe of the model, to which γ then makes essential reference, for example the familiar star-semantics for negation.

$$\models^{\mathfrak{M}}_{x} \neg\alpha \Leftrightarrow \not\models^{\mathfrak{M}}_{x*} \alpha$$

Critics of such strategies complain that although such strategies non-trivially realize sentences of the form $\alpha \wedge \neg\alpha$, they thereby make $\neg(\wedge)$ a mere homonym of the classical negator (conjunctor).

Others have added preservational features to $\models$, requiring it to preserve more than truth. That is, they specify certain properties ϕ of a set Σ (perhaps the value of some coherence-measure $\mu(\Sigma)$ for Σ) and require that if $\Sigma \models \alpha$, then that $\mu(\Sigma) \leq \mu(\Sigma \cup \alpha)$. See, for example, [16], [9], and [1]. But since the properties that they propose the $\models$ should preserve are undefined or constant for $\emptyset$, the strategy has no systemically distinct expression for the case of valid wffs. And again, these strategies agree with propositional logic on inferences from singletons. Accordingly they must distinguish between the pair $\{p, \neg p\}$ and , $\{p \wedge \neg p\}$. It follows that they can find expression only in implicational systems in which conjunction is non-classical.

Could there be a specifically preservationist approach to implication. Certainly the various variable-sharing requirements of relevant logics could be considered preservationist in spirit. In some vague but comforting way it might be taken to reflect the intuition that an implication requires that the consequent be about some feature of the world of which the antecedent is about some subfeature or vice versa. Implication then is required to preserve a kind of aboutness. The analytical implication of W.T. Parry [11] would be an early example.

A separate preservationist strategy first proposed in [7] and then more fully and systematically explored in [15] and [12] takes a different tack. It focuses upon the character of truth and its preservation. Properties have properties and properties of properties have properties. Whatever sort of property truth is, it presumably has properties and its properties have properties. In [7] truth-values of atoms are taken to be either fixed or unfixed, those of molecular wffs as fixed if no change in the truth-value of an atom will change its truth-value, unfixed else. This yields a 4-valued semantics that distinguishes paradoxes from non-paradoxical contradictions. The implication is required to preserve both truth and fixity. In the other work cited, this idea is extended to an infinite hierarchy by a generalization of Jaśkowski's Γ function (for an account of which *vide* [4].)

Here we consider a third, deeper but related alternative, one that, as we shall see, is not new except perhaps in this interpretation. We adjust $\models^{\mathfrak{M}}_{x}$, not by adding preservational restrictions to $\models$, or preservational requirements on $\rightarrow$ but by genetically so altering $\models$ that even for atomic wffs it is non-primitive: that is, it does more than distinguish 1 from 0. To put the matter another way, we take the usual account to embody what could be called an *unanalysed* truth-theory, and consider the prospects for an *analysed* truth-theory. We propose no particular analysis of truth, indeed profess no reason to suppose that there is any single such analysis that will do for all sentences from those about unanalysed medium-sized objects to those about quantum phenomena.

4. Taking truth seriously

No preservationist has yet proposed a system of inference the $\vdash$ of which did not preserve unanalysed truth. On any preservationist scheme ever proposed, if every wff of Σ is assigned a designated value and $\Sigma \vdash \alpha$, then α receives a designated value. Our reservation is as to whether that requirement, that way stated, is sufficiently rich to capture any metaphysically satisfying conception of the preservation of truth. Even a eulogist at a funeral would find more to say than that. So presumably more will be required for a resurrection. Does it not behove us to work up an account of truth that will take us beyond the featureless conversational notion and the bare mathematical recognition that whatever else they are, truth and falsity are distinct items? Sentences are composite objects: if they are to be true, then which parts they have in which order must play some role in determining whether they are true or false. So even for the simple sentences of ordinary life, truth must consist in some composition of features. And in the case of physical-theoretical sentences, to which in working life we never apply the language of truth without reservation, the case must be even more difficult to make out: a sentence's lease in a theory has at best an indefinite term; it runs from observation to observation, and can be terminated without notice whenever a more desirable tenant appears, or whenever what it pleases the landlord to call *the edifice* is renovated. We could not find out which of its empirical sentences were genuinely true at any finite stage in the history of a physical theory.

4.1. **Leibnizian truth and Meyer logics.** Leibniz supposed that all truth was analytic truth: but that the truth of those sentences called logically true is discoverable by finite analysis, while that of those thought of as empirically true requires an infinite analysis of which only God is capable. We make no such adventurous claim here. We suppose only that the notion of truth is infinitely rich, and that in consequence the truth-conditions of sentences have infinitely many bits to them. No one need disagree with this: the usual representation can be recovered by supposing that the infinitely many bits are indistinguishable. So we might have called our account 'leibnizian', not 'Leibnizian', naming it not after Gottfried Wilhelm Freiherr von Leibniz, but after Kleine Buchstaben von Leibniz, the one for whom the leibnizian truth-condition for necessity is named.[1] It will be evident in the sequel that such models might more descriptively be called Cantor models[2], as atoms find their values in the space of denumerable binary sequences.

Definition 1 (Cantor Models)**.** A Cantor propositional model is a pair $\langle U, V\rangle$ where U, (the *universe* of the model) is a non-empty set, and V is a *hyper-assignment*, that is, a function from $U\times$ At (At is the set of *atomic wffs*) to 2^N.That is, V assigns to each state/atom pair, x, i a function $V_{(x,i)} : N \to \{0, 1\}$.

Intuitively, the hyper-assignment tells us, for each atom p_i and each object x, which subset of a countably infinite set of tests the atom would pass at that object. The atom p_i would pass test j at x if $V_{(x,i)}(j) = 1$; p_i would not pass test j else. The conditional tense reflects the notion that no actual sequence of experimental tests is envisaged, merely that in order for an atom to be true, it must meet a set of conditions.

Of course this does not yet give us a truth-theory, except perhaps for sentences of the metalanguage asserting that an object-language sentence satisfies a condition, and thus would pass a test. But bear in mind that tests would be passed at objects, the natures of which have not been given. The most we can say of them is that for the language in question, an object in U is expected to provide at most the resources for a countably infinite set of hypothetical tests. So if the leibnizian approach seems to find after all an application for unanalysed truth, it is one that we understand only schematically. And we need make no assumption (though here we do so for simplicity's sake) that the objects do provide such resources. Nothing we have said so far settles the question as to what would count for an atom to pass a test, nor what for a non-atomic wff. Nor does anything we have said so far dictate the relationship between not passing a test, and failing it, or between the failure of p_i and the success of $\neg p_i$. Leibnizian truth-theory may be atomically gappy, though, again, the connection between atomic and molecular gappiness is so far unspecified.

[1]Michal Arciszewski, reminded us of this historical echo.

[2]The label was suggested by a remark of Oliver Schulte's.

As we have mentioned, we assume, for the sake of simplicity, that the pass/fail conditions for propositional wffs mimics the standard unanalysed truth-conditions. So $V_{(x,i)}$ is extended to a hyper-valuation for Φ (the set of propositional wffs) by:

$$V_{(x,i)}(\neg\alpha) = 1 - V_{(x,i)}(\alpha)$$
$$V_{(x,i)}(\alpha \to \beta) = Max(1 - V_{(x,i)}(\alpha), V_{(x,i)}(\beta))$$

The truth-theoretic question is this: how many and which tests must a wff pass in order to count as true?

In the simplest cases,

$$\models^{\mathfrak{M}}_{x} \alpha \text{ if } \forall i \in N, V_{(x,i)}(\alpha) = 1;$$
$$\not\models^{\mathfrak{M}}_{x} \alpha \; else.$$

Evidently on such a truth-theory, truth can be represented as S5 necessity: gappy but not paraconsistent. For some $\{\alpha, \neg\alpha\}$ pairs, neither will pass all of the tests at x, but for no such pairs will both pass every test. On the other hand, at the other extreme, if the truth-theory is such as to give

$$\models^{\mathfrak{M}}_{x} \alpha \text{ if } \forall i \in N : V_{(x,i)}(\alpha) = 1;$$
$$\not\models^{\mathfrak{M}}_{x} \alpha \; else.$$

then truth can be represented as $S5$ possibility: paraconsistent but not gappy. For some $\{\alpha, \neg\alpha\}$ pairs, both wffs will pass sufficiently many of the tests at x, but for no such pairs will neither pass sufficiently many. This modal logical connection is also a Jaśkowski connection [5]. Although $\{\alpha, \neg\alpha\}$ is an inconsistent set the closure of which is trivial, in $S5$, $\{\Diamond\alpha, \Diamond\neg\alpha\}$ is classically consistent. Thus a codification of the closure conditions on a set of $S5$ possibilities codifies a paraconsistent inference relation. Evidently such an inference relation does not admit $\wedge$-introduction, the $\Diamond$ of $S5$ being non-aggregative. In this respect, Jaśkowski's approach to paraconsistency is akin to that of some of the preservationist proposals. Apart from the major differences of detail, the larger scale difference between the two approaches is that for Jaśkowski, the non-aggregativity of the *S5* $\Diamond$ and the consequent exclusion of $\wedge$-introduction are precisely the appeal, whereas for the preservationists, it is typically some preservational requirement that defeats $\wedge$-introduction; it is by no means a grand strategy of the approach. For Jaśkowski it is itself a strategy; for the preservationists it is the consequence of a preservational requirement.

We have said by now enough to reveal that this territory is not completely unfamiliar historically, and other historically canvassed cases will have come to mind.

Definition 2 (truth-profiles)**.** $[\![\, \alpha \,]\!]^{\mathfrak{M}}_{x}$ (the profile of α at x in $\mathfrak{M}$) is $i \in \mathbf{N} | V_{(x,i)}(\alpha) = 1$.

Then the truth-theories we have considered can be given as:

$$[1] \models^{\mathfrak{M}}_{x} \alpha \Leftrightarrow [\![\, \alpha \,]\!]^{\mathfrak{M}}_{x} = \mathbf{N};$$

and

$$[2] \models^{\mathfrak{M}}_{x} \alpha \Leftrightarrow [\![\, \alpha \,]\!]^{\mathfrak{M}}_{x} \neq \emptyset.$$

but a truth-theory could plausibly be labelled *intuitionistic* (again lower-case) if it adopted as its standard:

$$[3] \models_x^{\mathfrak{M}} \alpha \Leftrightarrow [\![\alpha]\!]_x^{\mathfrak{M}} \text{is convex and cofinite}$$

since by this standard, α is true iff α is such that it would pass some test or other and thereafter pass every test. And, one might be tempted to say, so on. Leibnizian truth looks like leibnizian necessity, and the generalizations look like the generalizations of leibnizian necessity with necessity masquerading as truth, and truth masquerading as the passing of tests. We will say more later both about the air of familiarity and the persistent whiff of modality. For the moment we come to rest with a simple truth-conditional profile that does not on its face look like a disguised form of any independently known necessity. It is a particular instance of what might be called a *practical truth-theory*, one that recognizes the imperfection, perhaps the imperfectability, of working theoretical languages. It treats a sentence of a theoretical language as true, as it were, *for practical purposes*, if it satisfies sufficiently many sufficiently important conditions. We consider here an idealized and simplified form of such a truth-theory in which all that truth demands is a sufficient accumulation of satisfied conditions, more particularly, in this case,

$$[4] \models_x^{\mathfrak{M}} \alpha \Leftrightarrow [\![\alpha]\!]_x^{\mathfrak{M}} \text{is infinite.}$$

Observe first of all that on such a truth-theory the set of valid formulae, that is, the set of wffs true at every point in every model, is the set of classical tautologies. Thus this sort of truth is systemically classical, and therefore, for example, all instances of $p \to (\neg p \to q)$ are valid and *modus ponens* preserves validity. Nevertheless, the associated entailment is paraconsistent, since, as a moment's reflection reveals, *modus ponens*, like other non-trivially multi-premiss rules, does not in general preserve this kind of analysed truth. Neither, for example, does the system admit unrestricted $\wedge$-introduction. Now a system, one for which *mp* preserves validity, but not truth, is called a *Meyer system*.[3] So we can call the logic for the truth-theory [4] MPL, for Meyerized PL.

A word or two *de modo ponendo*. Corresponding to the preservation of validity is the proof-theoretic principle that MPL-provable implications preserve MPL-provability. That is, the condition [mp]

$$\frac{\alpha, \alpha \to \beta}{\beta}$$

is a closure condition on the set of theorems. But corresponding to the semantic feature that that condition is not a closure condition on the set of truths, is the proof-theoretic restriction that *mp* cannot be invoked unrestrictedly in MPL proofs. As a closure condition on MPL-theorems, the condition [mp] is the condition that provable implication preserves provability. But observe that valid implication preserves truth as well as validity, and correspondingly, a weakened form of *mp*, one citing $\alpha \to \beta$ as the last line of a proof from $\emptyset$ can justify a line of an MPL-proof. Furthermore, semantically, true implication preserves the truth of validities. So if $\models \alpha$ and $\models_x^{\mathfrak{M}} \alpha \to \beta$, then $\models_x^{\mathfrak{M}} \beta$, and correspondingly, weakened *mp* can justify β as a line, if the α line input ends a proof of α from $\emptyset$. But a set of tests that alternately favoured $\alpha \wedge \neg\beta$ and $\neg\alpha \wedge \neg\beta$, would make α true, and would make $\alpha \to \beta$ true, but would not make β true.

[3]So called for Bob Meyer, who professes not to like *modus ponens*.

A similar fate meets other multiple-premiss rules in MPL. The truth-theory is importantly holistic: pigeon-holistic. It admits all and only such multiple-premiss rules as are forced by pigeon-hole arguments. The upshot is that MPL takes us as near as damn to inferential explosions, but always denies us the final inferential ingredient.

The lesson is this: with a reformed truth-theory, even propositional logic is paraconsistent. On this simple diagnosis of classical propositional logic, the fault lies not in the interpretation of the connectives, but in the place of the interpretation within a theory of truth.

Theorem 1 (The fundamental theorem for MPL)**.** Every PL-consistent wff is true in a Cantor model.

Proof. Let α be a PL-consistent wff. Let $E = \{\beta_1, \ldots, \beta_i, \ldots\}$ be an enumeration of the set Φ of wffs, and Σ_0 be the maximal consistent extension of α along E. Now let $\gamma_1, \ldots, \gamma_i, \ldots$ be the natural ordering, induced by E, of the set $\{\alpha \wedge \beta \mid \beta \in \Phi \& \{\alpha, \beta\} \nvdash \bot\}$. Let Σ_i be the maximal consistent extension of γ_i along E.

Let $\mathfrak{M} = \langle \{\Sigma_i\}, V \rangle$, where V, the hyper-assignment, constantly assigns to each Σ_i, p_j pair, the function $V_{\{i,j\}}$, defined by

$$\begin{aligned} \forall i, j, V_{\{i,j\}} &= 1 \text{ if } p_j \in \Sigma_i; \\ &= 0 \text{ else.} \end{aligned}$$

Evidently, $\forall \beta \in \Phi,\ \forall_{i=0}^{\infty} \Sigma_i, \models^{\mathfrak{M}}_{\Sigma_i} \beta \iff [\![\ \beta\ ]\!]^{\mathfrak{M}}_x$ is infinite. In particular, $\forall_{i=0}^{\infty} \Sigma_i, \models^{\mathfrak{M}}_{\Sigma_i} \alpha$. □

4.2. **The modal logical connexion.** The truth-theory of [4] alone among those so far cited has no obviously modal inspiration. However, that truth-theory is equivalent to one according to which the natural order of the set of conditions on truth is taken to be an ordering of importance and given alethic significance, as in

$$[5] \models^{\mathfrak{M}}_x \alpha \Leftrightarrow \forall y \notin [\![\ \alpha\ ]\!]^{\mathfrak{M}}_x, \exists z \in [\![\ \alpha\ ]\!]^{\mathfrak{M}}_x : y < z.$$

Once again, truth is paraconsistent; validity is not, but multiple premiss rules cannot unrestrictedly be brought to bear inferentially, since they do not preserve truth. In fact the logic corresponding to this truth-theory is MPL.

But [5] once again wears the look, or at least the smell, of modality. In fact it apes the semantic idiom of natural frames:

$$[6] \models^{\mathfrak{M}}_x \Box\alpha \Leftrightarrow \forall y \notin [\![\ \alpha\ ]\!]^{\mathfrak{M}}, \exists z \in [\![\ \alpha\ ]\!]^{\mathfrak{M}} : y <_x z$$

the set of validities of which is completely axiomatized by the weak self-dual system that has elsewhere been labelled *SCon* ('S' for 'Segerberg'; For an account, see [6].)

[N]	$\vdash \Box\top$
[RM]	$\vdash \alpha \to \beta \Rightarrow \vdash \Box\alpha \to \Box\beta$
[Con]	$\vdash \neg\Box\bot.$

Now one reading of [5] has as the requirement for the truth of a sentence that for every test that it fails, there is a more significant one that it passes, as [6] has as the requirement for (deontic) necessity of a sentence that for every state in which it fails to obtain, there is a better one in which it does. It suggests as a reading for

$\models^{\mathfrak{M}}_{x} \alpha$, that α *ought to be regarded as true.* Lou Goble [2] has independently presented the same modal system in a related semantic idiom that adopts the truth-condition

$$[8] \models^{\mathfrak{M}}_{x} \Box\alpha \Leftrightarrow \exists y \in [\![\alpha]\!]^{\mathfrak{M}} \ \& \ \forall z, y <_x z \Rightarrow z \in [\![\alpha]\!]^{\mathfrak{M}}.$$

The truth-theory of

$$[9] \models^{\mathfrak{M}}_{x} \alpha \Leftrightarrow \exists y \in [\![\alpha]\!]^{\mathfrak{M}}_{x} \ \& \ \forall z, y <_x z \Rightarrow z \in [\![\alpha]\!]^{\mathfrak{M}}_{x}.$$

which is to [8] as [6] is to [7], is apparently stricter than that of [6], in that it requires eventual convexity of $[\![\alpha]\!]^{\mathfrak{M}}$, nevertheless corresponds also to MPL. The pigeon-hole strictures of the earlier discussion apply here also, save only to the non-convex portion of $[\![\alpha]\!]^{\mathfrak{M}}$.

Again the logic of *SCon* is the intersection of the logics of the *KnCon* systems of weakly aggregative deontic logics. These are the logics which adjoin to *SCon* instances of the schema

$$[K_n] \ \Box p_1 \wedge \cdots \wedge \Box p_{n+1} \rightarrow \Box \tfrac{2}{n+1}(p_1, \ldots, p_{n+1})$$

where $\frac{2}{n+1}(p_1, \ldots, p_{n+1})$ is the disjunction of all pairwise conjunctions of p_i, $p_j (1 \leqslant i \neq j \leqslant n+1)$. The interest of this family of logics is that the system *Kn* is completely axiomatized by the specification of a paraconsistent closure condition, n-forcing, on necessities.

$$\Sigma \Vdash_n \alpha \Leftrightarrow \forall \pi \in \Pi_n(\Sigma), \exists c \in \pi : c \vdash \alpha.$$

Thus, system *SCon* is therefore completely axiomatizable by the specification that, for every n, the set of necessities is closed under n-forcing, together with the postulate that $\bot$ is not a necessity.

5. The Moral

We understand *necessity* no better than we understand truth. *Necessarily alpha* is therefore a reading for $\Box\alpha$, and not an interpretation of it. In any of the uncountably many modal logics lacking the principle [T] $\Box\alpha \rightarrow \alpha$, we do not even have the excuse that that fundamental assumption of alethic necessity is satisfied, but needs must apply other Hellenic modifiers (*deontic, doxastic, stochastic* and so on) to suit the case, but we have nothing but intuition to tell us which principles distinguish the readings, nor which of the principles that a given semantic idiom imposes are principles that are wanted. In general we have no grounds for strong convictions about how the various semantic idiomata should be understood. One should, therefore, feel free to challenge the labels under which these systems are marketed. If systems that include all instances of [T] are to be called systems of (kinds of) necessary truth, why should systems lacking [T] be referred to as weak systems of necessity rather than as weak systems of (kinds of) truth?

Evidently a very strong paraconsistent propositional system is representable in the modal system *SCon*. Evidently, stronger such systems are representable in its *KnCon* extensions, stronger, that is, in admitting mixed and/or-introductions that *SCon* does not allow. And again, evidently, each such paraconsistent system can be thought of as introducing a distinctive theory of analysed truth, not, to be sure, one representable by hyper-assignments to $2^{\mathbf{N}}$, but an analysed truth-theory nevertheless.

Now all these technicalia will be of interest, no doubt, to those who find this sort of thing interesting. But it all prompts some observations both skeptical and hopeful. The skeptical observation is that we understand the notion of necessity no better than we understand the notion of truth. In fact, the Lady Chapel of alethic modal logic harbours all of the theological dustbunnies that blow about the larger shrine. The hopeful observations are, first, that the formal study of modal logic also comprises logics that are not alethic. Secondly, that therefore in studying the formal semantics of a mathematically abstract notion of necessity through the study of frame theory, formalists have been studying at a useful level of abstraction, the mathematical structure of theories of truth.

We can revisit the classic representation theorems in the light of this observation: Gödel's representation of Intuitionistic logic in *S4* and Goldblatt's representation (in [3]) of orthologic in Brouwersche modal logic can be understood as demonstrations of the mathematical structures of the truth-theories appropriate to the intended domains of application: constructive mathematics on the one hand, and quantum physics on the other. That an atom p_i of intuitionistic logic should require for the representation a translation into $\Box q_i$, that an atom of orthologic should require translation into $\Box\Diamond q_i$ can be interpreted as evidence that the atoms of some languages, perhaps the very languages where the dialetheists hoped for examples of mathematically or physically realized contradictions, are capable of being understood within particular analysed truth-theories.

We can also revisit the *Kn* systems. There we will now discover paraconsistent implicational logics, where before we found only modal representations of the familiar illative systems of forcing.

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