MODAL LOGICS FOR PRODUCTS OF TOPOLOGIES

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Abstract. We introduce the horizontal and vertical topologies on the product of topological spaces, and study their relationship with the standard product topology. We show that the modal logic of products of topological spaces with horizontal and vertical topologies is the fusion **S4** $\oplus$ **S4**. We axiomatize the modal logic of products of topological spaces with horizontal, vertical, and standard product topologies. We prove that both of these logics are complete for the product of rational numbers $\mathbb{Q} \times \mathbb{Q}$ with the appropriate topologies.

1. Introduction

The study of products of Kripke frames and their modal logics was initiated by Shehtman [17]. A systematic study of multi-dimensional modal logics of products of Kripke frames can be found in Gabbay and Shehtman [8], and for an up to date account of the most important results in the field we refer to Gabbay et al. [9]. We recall that for given two frames $\mathcal{F} = \langle W, S \rangle$ and $\mathcal{G} = \langle V, T \rangle$, the 'horizontal' and 'vertical' relations on the product $W \times V$ are defined as follows.

$$(w, v)R_1(w', v') \text{ iff } wSw' \text{ and } v = v'$$
$$(w, v)R_2(w', v') \text{ iff } w = w' \text{ and } vTv'$$

Amongst many other results, Gabbay and Shehtman proved that if L_1 and L_2 are modal logics complete with respect to frame classes $\mathbb{F}_1$ and $\mathbb{F}_2$ defined by universal Horn conditions and closed under taking disjoint unions, then the logic $L_1 \times L_2$ of the class of products

$$\mathbb{F}_1 \times \mathbb{F}_2 = \{\langle W \times V, R_1, R_2 \rangle : \langle W, S \rangle \in \mathbb{F}_1 \text{ and } \langle V, T \rangle \in \mathbb{F}_2\}$$

is axiomatized by the fusion $L_1 \oplus L_2$ plus the two additional principles of commutation $com = \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$ and convergence (also known as the Church-Rosser principle) $chr = \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$. In particular, since **S4** is complete with respect to the universal Horn class of reflexive and transitive frames, the product **S4** $\times$ **S4** is axiomatized as **S4** $\oplus$ **S4** plus *com* and *chr*.

It is known that topological semantics generalizes Kripke semantics for **S4**. In this paper we consider products of topological spaces. We generalize the notions of horizontal and vertical relations to horizontal and vertical topologies and study their relationship with the standard product topology. We show that the modal logic of products of topological spaces with horizontal and vertical topologies is **S4**$\oplus$**S4**, and the interaction principles *com* and *chr* only become valid when further restrictions are made on the topological spaces under consideration.

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Since the topological setting strongly suggests adding the 'true product topology', we also investigate the modal logic of products of topological spaces with all three topologies: horizontal, vertical, and the standard product topology. We show that the modal operator associated with the product topology is not definable in terms of the modal operators associated with the horizontal and vertical topologies, and we axiomatize the modal logic of products of topological spaces with all three topologies.

The paper is organized as follows. In Section 2 we recall some basic facts about topological semantics of **S4** and present a new proof of completeness of **S4** with respect to the rationals. We also review the definitions of the fusion $\mathbf{S4} \oplus \mathbf{S4}$ and the product $\mathbf{S4} \times \mathbf{S4}$. In Section 3 we introduce the horizontal and vertical topologies, and investigate their relationship with the standard product topology. Section 4 is concerned with the commutation and convergence principles in the topological setting, while Sections 5 and 6 contain completeness results for modal languages with operators corresponding to the horizontal, vertical, and standard product topologies. In the concluding Section 7 we point out some of the remaining open questions.

2. Preliminaries

2.1. **Topological completeness of S4.** If we interpret the modal operators $\Box$ and $\Diamond$ in topological spaces as the interior and closure operators, then the complete modal logic of all topological spaces is **S4** (McKinsey and Tarski [14]). A much stronger result, also due to McKinsey and Tarski, states that **S4** is in fact the complete modal logic of any metric separable dense-in-itself space. In particular, **S4** is the complete modal logic of the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, or the Cantor space $\mathbf{C}$. An alternative proof of completeness of **S4** with respect to $\mathbf{C}$ can be found in [15], and that with respect to $\mathbb{R}$ in [2]. In the subsequent sections we will need completeness of **S4** with respect to $\mathbb{Q}$. In order to make the paper self-contained, we present here an alternative proof of this fact, which might be of an independent interest.

To this end, recall that a topological space is a structure $\langle X, \tau \rangle$ where $\tau \subseteq \wp(X)$ contains $\emptyset$ and W and is closed under arbitrary unions and finite intersections. The elements of τ are called *open sets*. If, in addition, τ is closed under arbitrary intersections, then $\langle X, \tau \rangle$ is said to be *Alexandroff*. A topological model is a structure $M = \langle X, \tau, \nu \rangle$, where $\langle X, \tau \rangle$ is a topological space and ν is a valuation assigning subsets of X to propositional variables of the modal language. Then for $x \in X$, the modal operators $\Box$ and $\Diamond$ are interpreted as follows.

$$\begin{array}{lll} x \models \Box\varphi & \text{iff} & \exists U \in \tau : x \in U \text{ and } \forall y \in U (y \models \varphi) \\ x \models \Diamond\varphi & \text{iff} & \forall U \in \tau : \text{if } x \in U \text{ then } \exists y \in U (y \models \varphi) \end{array}$$

A *topo-bisimulation* between two topological models $M = \langle X, \tau, \nu \rangle$ and $M' = \langle X', \tau', \nu' \rangle$ is a non-empty relation $\leftrightharpoons \subseteq X \times X'$ such that if $x \leftrightharpoons x'$ then

(I) Base: $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any propositional variable p

(II) Forth condition: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ such that $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \leftrightharpoons y'$

(III) Back condition: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ such that $x \in U$ and for every $y \in U$ there is $y' \in U'$ with $y \leftrightharpoons y'$

An important feature of topo-bisimulations that will be used throughout is that they preserve truth of modal formulas [1].

Let T_2 be the infinite binary tree with the (reflexive and transitive) descendant relation. Formally, T_2 can be defined as $\langle W, R\rangle$, where $W = \{0,1\}^*$ is the set of finite strings (including the empty string) over $\{0,1\}$ and sRt iff $\exists u : s \cdot u = t$.

In our proof of completeness we will rely on the following two well-known results.

Theorem 2.1. *(van Benthem-Gabbay)* **S4** *is complete with respect to* T_2.

Proof. For a proof see, e.g., [11, Theorem 1 and the subsequent discussion]. The proof uses the fact that every finite rooted **S4**-frame is a bounded morphic image of T_2. □

Theorem 2.2. *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to* $\mathbb{Q}$.

Proof. For a proof see, e.g., [13, Page 217, Theorem 2]. □

Remark 2.3. We recall that if $\langle X, <\rangle$ is a linearly ordered set and $x, y \in X$ with $x < y$, then the *open interval* (x, y) is defined as the set $\{z \in X : x < z < y\}$. If we view linearly ordered sets as topological spaces using the set of open intervals as a basis for the topology, then it follows from Cantor's theorem that every countable dense linear ordering without endpoints is (as a topological space) homeomorphic to $\mathbb{Q}$.

We are now ready to proceed with the proof.

Theorem 2.4. **S4** *is complete with respect to* $\mathbb{Q}$.

Proof. As we pointed out earlier, this result is a particular case of the McKinsey and Tarski theorem [14]. An alternative proof can be extracted from [2]. Here we give yet another proof of this result, which we will build on in Sections 5 and 6 to obtain our two main completeness results.

Our strategy is as follows. We use completeness of **S4** with respect to T_2, view T_2 as an Alexandroff space, define a dense subset X of $\mathbb{Q}$ without endpoints, and establish a topo-bisimulation between X and T_2. This will allow us to transfer counterexamples from T_2 to X, which by Cantor's theorem is order-isomorphic, and hence homeomorphic to $\mathbb{Q}$.

Let $X = \bigcup_{n\in\omega} X_n$, where $X_0 = \{0\}$ and

$$X_{n+1} = X_n \cup \{x - \frac{1}{3^n}, x + \frac{1}{3^n} \mid x \in X_n\}$$

Claim 2.5. For $n > 0$ and $x, y \in X_n$, $x \neq y$ implies $|x - y| \geq \frac{1}{3^{n-1}}$.

Proof. By induction on n. If $n = 1$, then $X_1 = \{0, 1, -1\}$, and so $x \neq y$ implies $|x - y| \geq 1$. That the claim holds for $n = k + 1$ is also not hard to see. Note that if $u, v \in X_{n-1}$ with $u \neq v$, then, by induction hypothesis, $|u - v| \geq \frac{1}{3^{n-2}}$ and hence $|(u + \frac{1}{3^{n-1}}) - (v - \frac{1}{3^{n-1}})| \geq \frac{1}{3^{n-1}}$. □

It follows from Claim 2.5 that $\langle X, <\rangle$ is a countable dense linear ordering without endpoints, thus order-isomorphic, and hence homeomorphic to $\mathbb{Q}$. It also follows that for each $x \in X$ with $x \neq 0$ there exists n_x with $x \in X_{n_x}$ and $x \notin X_{n_x-1}$, and that there is a unique $y \in X_{n_x-1}$ with $x = y - \frac{1}{3^{n_x-1}}$ or $x = y + \frac{1}{3^{n_x-1}}$. Therefore, the open X-intervals $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ form a basis for the order-topology on X.

Now we define f from X onto T_2 by recursion (cf. Figure 4(a)): If $x = 0$ then we let $f(0)$ be the root r of T_2; if $x \neq 0$ then $x \in X_{n_x} - X_{n_x-1}$ and we let

$$f(x) = \begin{cases} \text{the left successor of } f(y) & \text{if } x = y - \frac{1}{3^{n_x-1}} \\ \text{the right successor of } f(y) & \text{if } x = y + \frac{1}{3^{n_x-1}} \end{cases}$$

Claim 2.6. f is open and continuous.

Proof. We recall that a basis for the Alexandroff topology on T_2 is $\mathcal{B} = \{B_t\}_{t\in T_2}$ where $B_t = \{s \in T_2 : tRs\}$. To show that f is open, for a basic open X-interval $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$, we show that $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)}$. Indeed, if $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ then $n_y > n_x$, and so $f(x)Rf(y)$. Conversely, if $f(x)Rt$ then it follows from the definition of f (by induction on the distance between $f(x)$ and t in the tree) that there exists $y \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ such that $f(y) = t$. Thus f is open.

To show that f is continuous it suffices to show that for each $t \in T_2$, the f-inverse image of B_t is open. Let $x \in f^{-1}(B_t)$. Then $tRf(x)$. So $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)} \subseteq B_t$. Thus there exists an open interval $I = (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ of x such that $I \subseteq f^{-1}(B_t)$, implying that f is continuous. □

To complete the proof, if $\mathbf{S4} \nvdash \varphi$, then by Theorem 2.1, there is a valuation ν on T_2 such that $\langle T_2, \nu\rangle, r \not\models \varphi$. Define a valuation ξ on X by $\xi(p) = f^{-1}(\nu(p))$. Since f is continuous and open and $f(0) = r$, we have that 0 and r are topo-bisimilar. Therefore, $\langle X, \xi\rangle, 0 \not\models \varphi$. Now since X is homeomorphic to $\mathbb{Q}$, we obtain that φ is also refutable on $\mathbb{Q}$. □

Note that the above completeness proof can also be seen as a representation argument. More precisely, we showed that every finite rooted **S4**-frame is a continuous and open image of $\mathbb{Q}$.

2.2. **The fusion S4 ⊕ S4.** Let $\mathcal{L}_{\Box_1\Box_2}$ be a bimodal language with modal operators $\Box_1$ and $\Box_2$. We recall that the *fusion* of **S4** with itself, denoted by **S4** ⊕ **S4**, is the least set of formulas containing **S4**-axioms for both $\Box_1$ and $\Box_2$, and closed under modus ponens, substitution, $\Box_1$-necessitation, and $\Box_2$-necessitation.

S4 ⊕ **S4**-frames are triples $\langle W, R_1, R_2\rangle$, where W is a nonempty set and R_1 and R_2 are reflexive and transitive. We call such a frame *rooted* if there is a $w \in W$ such that for all $v \in W$, it holds that $(w, v) \in (R_1 \cup R_2)^*$, where $(R_1 \cup R_2)^*$ is the reflexive transitive closure of $R_1 \cup R_2$.

Theorem 2.7. *(Kracht-Wolter and Fine-Schurz)* **S4** ⊕ **S4** *has the finite model property; in fact,* **S4** ⊕ **S4** *is complete with respect to finite rooted* **S4** ⊕ **S4***-frames.*

Proof. For a proof see, e.g., [9, Page 196, Theorem 4.2]. □

Let $T_{2,2}$ denote the infinite quaternary tree such that each node of $T_{2,2}$ is R_1-related to two of its four immediate successors and R_2-related to the other two; both R_1 and R_2 are taken to be reflexive and transitive. Formally $T_{2,2}$ can be defined as $\langle W, R_1, R_2\rangle$, where $W = \{0, 1, 2, 3\}^*$, sR_1t iff $\exists u \in \{0, 1\}^* : s \cdot u = t$, and sR_2t iff $\exists u \in \{2, 3\}^* : s \cdot u = t$ (see Figure 1). Clearly $T_{2,2}$ is a rooted **S4** ⊕ **S4**-frame.

Proposition 2.8. **S4** ⊕ **S4** *is complete with respect to* $T_{2,2}$*.*

Proof. A straightforward generalization of the standard unravelling procedure for **S4** (cf., e.g., [11] or [2]) unravels an arbitrary finite rooted **S4** ⊕ **S4**-frame into a bisimilar branching tree of the form $T_{2,2}$. For details see the forthcoming [16]. □

FIGURE 1. $T_{2,2}$. The solid lines represent R_1 and the dashed lines represent R_2. The dotted lines at the final nodes indicate that the pattern repeats on infinitely.

2.3. **The product S4 × S4.** For two **S4**-frames $\mathcal{F} = \langle W, S\rangle$ and $\mathcal{G} = \langle V, T\rangle$, define the *product frame* $\mathcal{F} \times \mathcal{G}$ to be the frame $\langle W \times V, R_1, R_2\rangle$, where for $w, w' \in W$ and $v, v' \in V$,

$$(w, v)R_1(w', v') \text{ iff } wSw' \text{ and } v = v'$$
$$(w, v)R_2(w', v') \text{ iff } w = w' \text{ and } vTv'$$

The frame $\mathcal{F}\times\mathcal{G}$ can be viewed as an **S4**⊕**S4**-frame by interpreting the modalities $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ as follows.

$$(w, v) \models \Box_1\varphi \quad \text{iff} \quad \forall(w', v') \text{ if } (w, v)R_1(w', v') \text{ then } (w', v') \models \varphi$$
$$(w, v) \models \Box_2\varphi \quad \text{iff} \quad \forall(w', v') \text{ if } (w, v)R_2(w', v') \text{ then } (w', v') \models \varphi$$

Let **S4** × **S4** denote the logic of products of **S4**-frames. As we pointed out in the introduction, the product logic **S4** × **S4** is axiomatized by adding the following two axioms to the fusion **S4** ⊕ **S4**:

$$com = \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$$

$$chr = \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$$

By the Sahlqvist theorem, *com* and *chr* have the following first-order correspondents:

$$\forall x\forall y(\exists z(xR_1z \wedge zR_2y) \leftrightarrow \exists z(xR_2z \wedge zR_1y))$$

$$\forall x\forall y\forall z((xR_1y \wedge xR_2z) \to \exists w(yR_2w \wedge zR_1w))$$

Besides R_1 and R_2, there is yet another (reflexive and transitive) relation on the product $W \times V$ defined componentwise:

$$(w, v)R(w', v') \text{ iff } wSw' \text{ and } vTv'$$

This allows us to interpret yet another modal operator $\Box$ in $\mathcal{F} \times \mathcal{G}$:

$$(w, v) \models \Box\varphi \quad \text{iff} \quad \forall(w', v') \text{ if } (w, v)R(w', v') \text{ then } (w', v') \models \varphi$$

However, since in product frames we have that $R = R_1 \circ R_2$, $\Box\varphi$ becomes equivalent to $\Box_1\Box_2\varphi$, and so $\Box$ turns out to be definable in terms of $\Box_1$ and $\Box_2$. As we will see shortly, in the subtler setting of topological products, the analogue of $\Box$ is not modally definable in terms of the analogues of $\Box_1$ and $\Box_2$.

3. Product spaces and product topo-bisimulations

3.1. **Horizontal and vertical topologies.** Let $\mathcal{X} = \langle X, \eta \rangle$ and $\mathcal{Y} = \langle Y, \theta \rangle$ be two topological spaces. Recall that the *standard product topology* τ on $X \times Y$ is defined by letting the sets $U \times V$ form a basis for τ, where U is open in $\mathcal{X}$ and V is open in $\mathcal{Y}$. Let I denote the interior operator and C denote the closure operator of τ.

We will define two additional one-dimensional topologies on $X \times Y$ by 'lifting' the topologies of the components.

Suppose $A \subseteq X \times Y$. We say that A is *horizontally open (H-open)* if for any $(x, y) \in A$ there exists $U \in \eta$ such that $x \in U$ and $U \times \{y\} \subseteq A$. Similarly, we say that A is *vertically open (V-open)* if for any $(x, y) \in A$ there exists $V \in \theta$ such that $y \in V$ and $\{x\} \times V \subseteq A$. If A is both H- and V-open, then we call it *HV-open*. H-closed, V-closed and HV-closed sets are defined similarly. Let τ_1 denote the set of all H-open subsets of $X \times Y$ and τ_2 denote the set of all V-open subsets of $X \times Y$. It is easy to verify that both τ_1 and τ_2 form topologies on $X \times Y$. We call τ_1 the *horizontal topology* and τ_2 the *vertical topology*. The closure and interior operators C_i and I_i for τ_i can be defined in the usual way ($i = 1, 2$).

Remark 3.1. It is obvious that a set open in the standard product topology is both horizontally and vertically open. That is $\tau \subseteq \tau_1$ and $\tau \subseteq \tau_2$. However, the converse inclusions don't hold in general. In fact, we will show below that I is not modally definable by means of I_1 and I_2.

The interpretation of the modal operators $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ in $\langle X \times Y, \tau_1, \tau_2 \rangle$ is as expected:

$$(x, y) \models \Box_1\varphi \text{ iff } (\exists U \in \tau_1)((x, y) \in U \text{ and } \forall (x', y') \in U \,.\, (x', y') \models \varphi)$$
$$(x, y) \models \Box_2\varphi \text{ iff } (\exists V \in \tau_2)((x, y) \in V \text{ and } \forall (x', y') \in V \,.\, (x', y') \models \varphi)$$

The modalities $\Diamond_1$ and $\Diamond_2$ are defined dually. Furthermore, all the usual notions, such as satisfiability and validity, generalize naturally to this new language.

The one-dimensional nature of the horizontal and vertical topologies is emphasized by the following proposition.

Proposition 3.2. (1) *A formula φ constructed from the Booleans and the modal operator $\Box_1$ is valid in $\langle X \times Y, \tau_1, \tau_2 \rangle$ iff φ is valid in $\langle X, \eta \rangle$.*
(2) *A formula φ constructed from the Booleans and the modal operator $\Box_2$ is valid in $\langle X \times Y, \tau_1, \tau_2 \rangle$ iff φ is valid in $\langle Y, \theta \rangle$.*

Proof. See the forthcoming [16] for details on this and similar results. □

One of the most prominent examples of a product of topological spaces is the real plane $\mathbb{R} \times \mathbb{R}$. With this example in mind, our choice to study the horizontal and vertical topologies and their interaction might seem poorly motivated. Indeed, if one chooses to consider these two topologies, then why not also all their linear combinations? And, more generally, are we not leaving the territory of pure topology by taking into account a limited form of metric information?

A first answer to this objection is that, while on the real plane the horizontal and vertical directions seem no more special than any other linear combination of them, this is an artifact of the particular space in question. When considering arbitrary topological spaces, the notion of "linear combinations" is no longer well defined. A second answer is that, when considering scenarios where the two topological spaces have sufficiently distinct roles (for example, in the spatio-temporal reasoning, as opposed to the two-dimensional spatial reasoning), the horizontal (i.e., temporal) and vertical (i.e., spatial) directions seem to have special status.

A different motivation for studying horizontal and vertical topologies on product spaces comes from modal logic. It is well known that topological semantics of modal logic generalizes relational semantics for normal extensions of **S4**. To be more specific, with every **S4**-frame $\langle W, R\rangle$ corresponds the topological space $\langle W, \tau_R\rangle$, where τ_R contains precisely the R-upward closed subsets of W. Now, for **S4**-frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{F}' = \langle W', R'\rangle$, let $\mathfrak{F} \times \mathfrak{F}' = \langle W \times W', R_1, R_2\rangle$ be their product, as defined in the introduction. Then τ_{R_1} and τ_{R_2} are precisely the horizontal and vertical topologies on the product space $W \times W'$. This shows that our topological product construction is a faithful generalization of the usual product construction for Kripke frames. Of course, it does not show that it is a *natural* generalization, but we believe that it is (even apart from the lack of other natural candidates). We will return to this point at the end of the paper.

3.2. **Failure of** ***com*** **and** ***chr*** **on $\mathbb{R} \times \mathbb{R}$.** We saw in the previous subsection that whenever topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are representable as **S4**-frames (are Alexandroff), then the horizontal and vertical topologies on their product $X \times Y$ can be defined from the horizontal and vertical relations on the product of these frames. In other words, our topological setting generalizes the case for products of Kripke frames. Nevertheless, there are crucial differences between these two settings. In particular, both *com* and *chr*, while valid on products of Kripke frames, can be refuted on topological products. To stimulate intuitions before plunging into general theory, we exhibit their failure on $\mathbb{R} \times \mathbb{R}$.

(a) Failure of *com*: Let

$$\nu(p) = \Big(\bigcup_{x\in(-1,0)} \{x\} \times (x, -x)\Big) \cup (\{0\} \times (-1, 1)) \cup \Big(\bigcup_{x\in(0,1)} \{x\} \times (-x, x)\Big)$$

(see Figure 2a). Then there is a basic horizontal open $(-1, 1) \times \{0\}$ such that $(0, 0)$ is in it and every point in $(-1, 1) \times \{0\}$ sits in a vertically open subset of p. Thus, $\Box_1\Box_2 p$ is true at $(0, 0)$. On the other hand, there is no vertical open containing $(0, 0)$ in which every point sits inside a horizontally open subset of p, implying that $\Box_2\Box_1 p$ is false at $(0, 0)$.

(b) Failure of *chr*: Let $\nu(p) = \bigcup\{\{\frac{1}{n}\} \times (-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ (see Figure 2b). Then in any basic horizontal open around $(0, 0)$ there is a point that sits in a basic vertical open in which p is true everywhere. Thus, $\Diamond_1\Box_2 p$ is true at $(0, 0)$. On the other hand, since the horizontal closure of $\nu(p)$ is $\nu(p) \cup \{(0, 0)\}$ and since the vertical interior of $\nu(p) \cup \{(0, 0)\}$ is $\nu(p)$, we have that $(0, 0)$ is not in $I_2(C_1(\nu(p)))$, implying that $\Box_2\Diamond_1 p$ is false at $(0, 0)$.

As we will see in Section 4, the structure of these counterexamples on $\mathbb{R} \times \mathbb{R}$ is not accidental. We will show under which circumstances they can be reproduced in other products of topological spaces.

3.3. **Product topo-bisimulations.** As in Kripke semantics, an appropriate notion of bisimulation plays crucial role in understanding and developing topological semantics. In this subsection we generalize the notion of topo-bisimulation introduced in Section 2.1 to topological models equipped with several topologies. We will

FIGURE 2. Counterexamples for *com* and *chr* on $\mathbb{R} \times \mathbb{R}$.

use it to show that the standard product interior is not definable in terms of the horizontal and vertical interiors. Another important application of multi-dimensional topo-bisimulations will come in the completeness proofs below.

We exhibit the case of two topologies, but the generalization to any number of topologies is straightforward.

Definition 3.3. Let $M = \langle X, \tau_1, \tau_2, \nu\rangle$ and $M' = \langle X', \tau'_1, \tau'_2, \nu'\rangle$ be topological models equipped with two topologies each. A 2-*topo-bisimulation* is a nonempty relation $\leftrightharpoons \subseteq X \times X'$ such that if $x \leftrightharpoons x'$ then the following hold for $i = 1, 2$:

(I) BASE: $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any proposition variable p

(II) FORTH CONDITION: $x \in U \in \tau_i$ implies that there exists $U' \in \tau'_i$ such that $x' \in U'$ and for all $z' \in U'$ there exists $z \in U$ with $z \rightleftharpoons z'$

(III) BACK CONDITION: $x' \in U' \in \tau'_i$ implies that there exists $U \in \tau_i$ such that $x \in U$ and for all $z \in U$ there exists $z' \in U'$ with $z \rightleftharpoons z'$

The 2-topo-bisimulation $\rightleftharpoons$ is called *total* if it is defined for all elements of X and X', i.e., $dom(\rightleftharpoons) = X$ and $rng(\rightleftharpoons) = X'$. The fundamental invariance property of 2-topo-bisimulations is given by the following proposition.

Proposition 3.4. *Let $M = \langle X, \tau_1, \tau_2, \nu\rangle$ and $M' = \langle X', \tau'_1, \tau'_2, \nu'\rangle$ be topological models equipped with two topologies each, and let $x \rightleftharpoons x'$ for some 2-topo-bisimulation $\rightleftharpoons \subseteq X \times X'$. Then for every modal formula φ in $\mathcal{L}_{\Box_1\Box_2}$ we have that $M, x \models \varphi$ iff $M', x' \models \varphi$.*

Proof. The proof is a straightforward generalization of the 1-topo-bisimulation version found in [1] and we omit the details of the induction. □

Definition 3.3 and Proposition 3.4 apply to arbitrary topological models M, M' with two or more topologies each. By analogy with Kripke semantics, one can think of such models as fusion models. In the special case when M and M' consist of product spaces with the horizontal and vertical topologies, the 2-topo-bisimulation $\rightleftharpoons$ is called a *product topo-bisimulation*.

Topo-bisimulations are useful for showing that properties are not definable in our language. A nice example of this is given in Proposition 3.5 below. For two topological spaces $\mathcal{X}$ and $\mathcal{Y}$, consider the product space $\langle X \times Y, \tau, \tau_1, \tau_2 \rangle$, where τ stands for the standard product topology, τ_1 for the horizontal topology, and τ_2 for the vertical topology. We recall that $\Box_1$ and $\Box_2$ are interpreted via the horizontal and vertical topologies, while $\Box$ is interpreted via the standard product topology.

Proposition 3.5. *$\Box$ is not definable in the language $\mathcal{L}_{\Box_1\Box_2}$.*

Proof. It is sufficient to find two product models $M = \langle X \times Y, \tau_1, \tau_2, \nu \rangle$ and $M' = \langle X' \times Y', \tau_1', \tau_2', \nu' \rangle$ with $(x, y) \in X \times Y$ and $(x', y') \in X' \times Y'$, and a product topo-bisimulation $\leftrightharpoons \subseteq (X \times Y) \times (X' \times Y')$ such that $(x, y) \leftrightharpoons (x', y')$, that $M, (x, y) \models \Diamond p$, and that $M', (x, y) \not\models \Diamond p$. Since all formulae in the language $\mathcal{L}_{\Box_1\Box_2}$ are preserved by product topo-bisimulations and $\Diamond p$ is not, we conclude that $\Diamond p$ is not equivalent to any formula of $\mathcal{L}_{\Box_1\Box_2}$ (or to any infinite conjunction of such formulae for that matter). It follows that neither is $\Box p$.

For the product space we take $\mathbb{Q} \times \mathbb{Q}$. Let $\nu(p) = \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ and $\nu'(p) = \emptyset$. Let also $\leftrightharpoons$ be the identity relation on $(\mathbb{Q} \times \mathbb{Q}) \setminus \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$. It is not hard to see that $\leftrightharpoons$ is a product topo-bisimulation between the models $\langle \mathbb{Q} \times \mathbb{Q}, \nu \rangle$ and $\langle \mathbb{Q} \times \mathbb{Q}, \nu' \rangle$ that connects $(0, 0)$ to $(0, 0)$. Since $(0, 0)$ is in the closure of $\nu(p)$, we have that $\langle \mathbb{Q} \times \mathbb{Q}, \nu \rangle, (0, 0) \models \Diamond p$. On the other hand, it is obvious that $\langle \mathbb{Q} \times \mathbb{Q}, \nu' \rangle, (0, 0) \models \Box \neg p$. $\Box$

Another example for the use of topo-bisimulations is Proposition 3.6. Given topological product spaces $\langle X \times Y, \tau_1, \tau_2 \rangle$ and $\langle X' \times Y', \tau_1', \tau_2' \rangle$, we say that a map $f : X \times Y \to X' \times Y'$ is *HV-continuous* if it is continuous with respect to both horizontal vertical topologies, and that f is *HV-open* if it is open withe respect to both topologies. HV-open HV-continuous bijections are called *HV-homeomorphisms*. Note that if X is homeomorphic to X' and Y is homeomorphic to Y', then $X \times Y$ is HV-homeomorphic to $X' \times Y'$.

Proposition 3.6. *Surjective HV-continuous HV-open maps preserve validity of formulas of $\mathcal{L}_{\Box_1\Box_2}$.*

Proof. Let $\langle X \times Y, \tau_1, \tau_2 \rangle$ and $\langle X' \times Y', \tau_1', \tau_2' \rangle$ be given, and let $f : X \times Y \to X' \times Y'$ be surjective, HV-continuous, and HV-open. For a valuation ν' on $X' \times Y'$ we can define a valuation ν on $X \times Y$ by putting $\nu(p) = f^{-1}(\nu'(p))$. Then it is easy to verify that f is a total 2-topo-bisimulation between the models $M = \langle X \times Y, \tau_1, \tau_2, \nu \rangle$ and $M' = \langle X' \times Y', \tau_1', \tau_2', \nu' \rangle$. It follows that whenever a formula of $\mathcal{L}_{\Box_1\Box_2}$ is refuted on the latter model, it can also be refuted on the former one. $\Box$

4. Correspondence for *com* and *chr*

As we have seen above, unlike products of Kripke frames, products of topological spaces do not always validate *com* and *chr*. In this section we specify those classes of products of topological spaces in which *com* and *chr* hold. We start by investigating the validity of *com*. It is useful to split *com* into $com_{\to} = \Box_1\Box_2 p \to \Box_2\Box_1 p$ and $com_{\leftarrow} = \Box_2\Box_1 p \to \Box_1\Box_2 p$.

Let $\mathcal{X} = \langle X, \eta \rangle$ be a topological space. We recall that $\mathcal{X}$ is Alexandroff if the intersection of any family of open sets is again open. We call $\mathcal{X}$ *κ-Alexandroff* if the intersection of any family of open sets of cardinality κ is again open; that is, $\eta' \subseteq \eta$ and $|\eta'| \leq \kappa$ imply $\bigcap \eta' \in \eta$.

Proposition 4.1. *If $\mathcal{X} = \langle X, \eta\rangle$ is κ-Alexandroff and $|Y| \leq \kappa$, then $\mathcal{X}\times\mathcal{Y} \models com_{\leftarrow}$ and $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$.*

Proof. We show that $\mathcal{X}\times\mathcal{Y} \models com_{\leftarrow}$. That $\mathcal{Y}\times\mathcal{X} \models com_{\rightarrow}$ is proved symmetrically. Suppose for a point $(x, y) \in X \times Y$ and a valuation ν on $\mathcal{X} \times \mathcal{Y}$ we have that $(x, y) \models \Box_2\Box_1 p$. Then there exists a neighborhood V of y such that for each $z \in V$ there is a neighborhood U_z of z with $U_z \times \{z\} \subseteq \nu(p)$. Since $|V| \leq \kappa$ and $\mathcal{X}$ is κ-Alexandroff, we have that $U = \bigcap\{U_z : z \in V\} \in \eta$. But then $U \times V \subseteq \nu(p)$, implying that $(x, y) \models \Box_1\Box_2 p$. □

Corollary 4.2. *If $\mathcal{X}$ is Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$ and $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$ for any topological space $\mathcal{Y}$.*

Proof. It is sufficient to observe that every Alexandroff space is κ-Alexandroff for every cardinal κ, and apply Proposition 4.1. □

It follows that if both $\mathcal{X}$ and $\mathcal{Y}$ are Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models com$. Given the well-known correspondence between Kripke frames for **S4** and Alexandroff topologies, the above corollary sheds some topological light on the validity of *com* on products of Kripke frames.

The converse of Corollary 4.2 does not hold. For instance, every topology commutes with the discrete topology of any cardinality. Thus, it can happen that $\mathcal{X}$ or $\mathcal{Y}$ are not Alexandroff and yet $\mathcal{X} \times \mathcal{Y} \models com$. However, if $\mathcal{X}$ and $\mathcal{Y}$ coincide, then the converse of Corollary 4.2 holds. To see this, for $x \in X$, let η_x denote the set of all neighborhoods of x.

Lemma 4.3. *If $\mathcal{X}$ is not Alexandroff, then there is a point $x \in X$ such that $\bigcap \eta_x \notin \eta$.*

Proof. Since $\mathcal{X}$ is not Alexandroff, there exists a set B of opens such that $\bigcap B \notin \eta$. Let $x \in \bigcap B$. Obviously $\bigcap \eta_x \subseteq \bigcap B$ and $\bigcap B = \bigcup\{\bigcap \eta_x : x \in \bigcap B\}$. If $\bigcap \eta_x$ were open for every $x \in \bigcap B$, then $\bigcap B$ would be open. Therefore, there exists $x \in \bigcap B$ such that $\bigcap \eta_x$ is not open. □

Proposition 4.4. *If $\mathcal{X}$ is not Alexandroff, then $\mathcal{X} \times \mathcal{X} \not\models com_{\leftarrow}$ and $\mathcal{X} \times \mathcal{X} \not\models com_{\rightarrow}$.*

Proof. We show that $\mathcal{X} \times \mathcal{X} \not\models com_{\leftarrow}$. The case for $\mathcal{X} \times \mathcal{X} \not\models com_{\rightarrow}$ is symmetric. Since $com_{\leftarrow}$ is equivalent to $\Diamond_1\Diamond_2 p \rightarrow \Diamond_2\Diamond_1 p$, it is enough to show that $\mathcal{X} \times \mathcal{X} \not\models \Diamond_1\Diamond_2 p \rightarrow \Diamond_2\Diamond_1 p$. As $\mathcal{X}$ is not Alexandroff, by Lemma 4.3 there exists $x \in X$ such that $\bigcap \eta_x \notin \eta$. Let $\eta_x = \{U_i\}_{i\in I}$. We order I by putting $i \leq j$ iff $U_i \supseteq U_j$. Since $U_i, U_j \in \eta_x$ implies $U_i \cap U_j \in \eta_x$, it follows that $(I, \leq)$ is a directed partial order. Let $J = \{i \in I : \exists j \geq i \text{ with } U_i - U_j \neq \emptyset\}$. We show that J is cofinal in I. If not, then there exists $i \in I$ such that for any $j \geq i$ we have $U_i - U_j = \emptyset$. Therefore, $U_i = U_j$ for any $j \geq i$. Thus, $\bigcap \eta_x = \bigcap_{i\in I} U_i = \bigcap_{j\geq i} U_i = U_i \in \eta$, a contradiction. For $i \in J$ let $j \geq i$ be such that $U_i - U_j \neq \emptyset$ and pick $x_i \in U_i - U_j$. Then $\{x_i\}_{i\in J}$ is a net converging to x. Let ν be a valuation on $\mathcal{X} \times \mathcal{X}$ such that $\nu(p) = \{(x_i, x_j) : i, j \in J \text{ and } i \leq j\}$. For $U \in \eta_x$ and $i \in J$, let $U_j = U \cap U_i$. Then $i \leq j$. Since J is cofinal in I we can assume that $j \in J$. Therefore, $(x_i, x_j) \in \nu(p)$. It follows that $(x_i, x) \models \Diamond_2 p$. Thus, $(x, x) \models \Diamond_1\Diamond_2 p$. On the other hand, for any $U \in \eta_x$ and for any $x_j \in U$ we have $(U_i \times \{x_j\}) \cap \nu(p) = \emptyset$ for any $i \in J$ with $i > j$. Therefore, $(x, x) \not\models \Diamond_2\Diamond_1 p$. □

From Corollary 4.2 and Proposition 4.4 we obtain the following characterization of Alexandroff spaces.

Corollary 4.5. *The following conditions are equivalent:*

(1) $\mathcal{X}$ *is Alexandroff.*
(2) $\mathcal{X} \times \mathcal{X} \models com$.
(3) $\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$ *for every topological space* $\mathcal{Y}$.
(4) $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$ *for every topological space* $\mathcal{Y}$.

We end this section by investigating validity of *chr* in the products of topological spaces.

Proposition 4.6. *If either* $\mathcal{X}$ *or* $\mathcal{Y}$ *is Alexandroff, then* $\mathcal{X} \times \mathcal{Y} \models chr$.

Proof. Let $\mathcal{X} = \langle X, \eta \rangle$ and $\mathcal{Y} = \langle Y, \theta \rangle$. First suppose that $\mathcal{X}$ is Alexandroff. So every $x \in X$ has a least neighborhood U_x. If for a valuation ν on $\mathcal{X} \times \mathcal{Y}$ and a point $(x, y) \in X \times Y$ we have that $(x, y) \models \Diamond_1 \Box_2 p$, then there exists $z \in U_x$ such that $(z, y) \models \Box_2 p$. Therefore, there exists $V \in \theta_y$ such that $\{z\} \times V \subseteq \nu(p)$. But then for every $u \in V$ we have $(x, u) \models \Diamond_1 p$, implying that $(x, y) \models \Box_2 \Diamond_1 p$.

Now suppose that $\mathcal{Y}$ is Alexandroff. So every $y \in Y$ has a least neighborhood V_y. If for a valuation ν on $\mathcal{X} \times \mathcal{Y}$ and a point $(x, y) \in X \times Y$ we have that $(x, y) \models \Diamond_1 \Box_2 p$, then for every $U \in \eta_x$ there exists $z \in U$ such that $\{z\} \times V_y \subseteq \nu(p)$. But then for every $u \in V_y$ and for every $U \in \eta_x$ there exists $z \in U$ such that $(z, u) \in \nu(p)$. Thus, $(x, y) \models \Box_2 \Diamond_1 p$. □

Since Kripke frames for **S4** correspond to Alexandroff topologies, the above proposition gives a topological insight into the soundness of *chr* with respect to products of Kripke frames. Even though the converse of Proposition 4.6 is not in general true, similar to the case with *com*, we have that if $\mathcal{X}$ and $\mathcal{Y}$ coincide, then the converse does indeed hold.

Proposition 4.7. *If* $\mathcal{X}$ *is not Alexandroff, then* $\mathcal{X} \times \mathcal{X} \not\models chr$.

Proof. Let $x \in X$, $\eta_x = \{U_i\}_{i \in I}$, $J \subseteq I$, and the net $\{x_i\}_{i \in J}$ be chosen as in the proof of Proposition 4.4. We define a valuation ν on $\mathcal{X} \times \mathcal{X}$ by putting $\nu(p) = \bigcup_{i \in J}(\{x_i\} \times U_i)$. Then it is easy to verify that $(x, x) \models \Diamond_1 \Box_2 p$ but $(x, x) \not\models \Box_2 \Diamond_1 p$. □

Propositions 4.6 and 4.7 lead to yet another characterization of Alexandroff spaces.

Corollary 4.8. *The four equivalent conditions in Corollary 4.5 are equivalent to the following one:*

(5) $\mathcal{X} \times \mathcal{X} \models chr$.

For more results in this direction we refer to the forthcoming [12].

5. The logic of product spaces

As we saw in the previous section, both *com* and *chr* can be refuted on products of topological spaces. This suggests that the complete logic of all products of topological spaces is weaker than **S4** × **S4**. The main goal of this section is to show that this logic is **S4** ⊕ **S4**. In fact, we will show that **S4** ⊕ **S4** is complete with respect to $\mathbb{Q} \times \mathbb{Q}$.

Theorem 5.1. **S4** $\oplus$ **S4** *is complete with respect to* $\mathbb{Q}\times\mathbb{Q}$.

Proof. By Proposition 2.8, **S4** $\oplus$ **S4** is complete with respect to the infinite quaternary tree $T_{2,2} = \langle W, R_1, R_2\rangle$. We view $T_{2,2}$ as equipped with two Alexandroff topologies defined from R_1 and R_2. To prove completeness of **S4** $\oplus$ **S4** with respect to $\mathbb{Q}\times\mathbb{Q}$ we take the X constructed in the proof of Theorem 2.4, define recursively a HV-open subspace Y of $X \times X$ and a continuous open map g from Y onto $T_{2,2}$ with respect to both topologies: this will allow us to transfer counterexamples from $T_{2,2}$ to Y, then from Y to $X \times X$, and finally from $X\times X$ to $\mathbb{Q}\times\mathbb{Q}$.

Let $Y = \bigcup_{n\in\omega} Y_n$, where $Y_0 = \{(0,0)\}$ and

$$Y_{n+1} \;=\; Y_n \cup \{(x-\frac{1}{3^n}, y), (x+\frac{1}{3^n}, y), (x, y-\frac{1}{3^n}), (x, y+\frac{1}{3^n}) \mid (x,y)\in Y_n\}$$

Claim 5.2. Y is a HV-open subspace of $X\times X$.

Proof. Let $(x,y)\in Y$. Then $x \in (x-\frac{1}{3^{n_x}}, x+\frac{1}{3^{n_x}}) \subseteq X$. Therefore, $(x,y)\in (x-\frac{1}{3^{n_x}}, x+\frac{1}{3^{n_x}})\times\{y\}\subseteq Y$. Thus, Y is a H-open subspace of $X\times X$. That Y is a V-open subspace of $X\times X$ is proved symmetrically. $\square$

A similar argument as before shows that for each $(x,y)\in Y$ such that $(x,y)\neq(0,0)$ there exists $n_{(x,y)}$ with $(x,y)\in Y_{n_{(x,y)}}$ and $(x,y)\notin Y_{n_{(x,y)}-1}$, and that there is a unique $(u,v)\in Y_{n_{(x,y)}-1}$ such that $(x,y) = (u\pm\frac{1}{3^{n_{(x,y)}-1}}, v)$ or $(x,y) = (u, v\pm\frac{1}{3^{n_{(x,y)}-1}})$.

We define g from Y onto $T_{2,2}$ by recursion (cf. Figure 4(b)): If $(x,y) = (0,0)$ then we let $g(0,0)$ be the root r of $T_{2,2}$; if $(x,y)\neq(0,0)$ then $(x,y) = (u\pm\frac{1}{3^{n_{(x,y)}-1}}, v)$ or $(x,y) = (u, v\pm\frac{1}{3^{n_{(x,y)}-1}})$ for a unique $(u,v)\in Y_{n_{(x,y)}-1}$, and we let

$$g(x,y) = \begin{cases} \text{the left } R_1\text{-successor of } g(u,v) & \text{if } (x,y) = (u-\frac{1}{3^{n_{(x,y)}-1}}, v)\\ \text{the right } R_1\text{-successor of } g(u,v) & \text{if } (x,y) = (u+\frac{1}{3^{n_{(x,y)}-1}}, v)\\ \text{the left } R_2\text{-successor of } g(u,v) & \text{if } (x,y) = (u, v-\frac{1}{3^{n_{(x,y)}-1}})\\ \text{the right } R_2\text{-successor of } g(u,v) & \text{if } (x,y) = (u, v+\frac{1}{3^{n_{(x,y)}-1}}) \end{cases}$$

Claim 5.3. g is open and continuous with respect to both topologies.

Proof. Let τ_1 and τ_2 denote the restrictions of the horizontal and vertical topologies of $X\times X$ to Y, respectively. We prove that g is open and continuous with respect to τ_1. That it is open and continuous with respect to τ_2 is proved symmetrically. We observe that

$$\{(x-\frac{1}{3^{n_{(x,y)}}}, x+\frac{1}{3^{n_{(x,y)}}})\times\{y\} | (x,y)\in Y\}$$

forms a basis for τ_1. We also recall that a basis for the Alexandroff topology on $T_{2,2}$ defined from R_1 is $\mathcal{B}_1 = \{B^1_t\}_{t\in T_{2,2}}$ where $B^1_t = \{s\in T_{2,2} : tR_1 s\}$.

To see that g is open, let $(x-\frac{1}{3^{n_{(x,y)}}}, x+\frac{1}{3^{n_{(x,y)}}})\times\{y\}$ be a basic open for τ_1. Then the same argument as in Claim 2.6 guarantees that $g((x-\frac{1}{3^{n_{(x,y)}}}, x+\frac{1}{3^{n_{(x,y)}}})\times\{y\}) = B^1_{g(x,y)}$. Thus g is open. To see that g is continuous it suffices to show that for each $t\in T_{2,2}$, the g-inverse image of B^1_t belongs to τ_1. Let $(x,y)\in g^{-1}(B^1_t)$. Then $tR_1 g(x,y)$. So $g((x-\frac{1}{3^{n_{(x,y)}}}, x+\frac{1}{3^{n_{(x,y)}}})\times\{y\}) = B^1_{g(x,y)}\subseteq B^1_t$. Thus there exists an open neighborhood $U = (x-\frac{1}{3^{n_{(x,y)}}}, x+\frac{1}{3^{n_{(x,y)}}})\times\{y\}$ of (x,y) such that $U\subseteq g^{-1}(B^1_t)$, implying that g is continuous. $\square$

To complete the proof, if $\mathbf{S4} \oplus \mathbf{S4} \not\vdash \varphi$, then by Proposition 2.8, there is a valuation ν on $T_{2,2}$ such that $\langle T_{2,2}, \nu\rangle, r \not\models \varphi$. Define a valuation ξ on Y by $\xi(p) = g^{-1}(\nu(p))$. Since g is continuous and open with respect to both topologies and $g(0,0) = r$, we have that $(0,0)$ and r are 2-topo-bisimilar. Therefore, $\langle Y, \xi\rangle, (0,0) \not\models \varphi$. Now since Y is a HV-open subset of $X \times X$, we obtain that φ is refutable on $X \times X$. Finally, Theorem 2.4 implies that X is homeomorphic to $\mathbb{Q}$. Therefore, $X \times X$ is HV-homeomorphic to $\mathbb{Q} \times \mathbb{Q}$, and hence φ is also refutable on $\mathbb{Q} \times \mathbb{Q}$. $\square$

Corollary 5.4. $\mathbf{S4} \oplus \mathbf{S4}$ *is the logic of products of arbitrary topologies.*

It follows that the logic of products of arbitrary topologies is decidable and has a $PSPACE$-complete satisfiability problem [18]. This stands in contrast with the satisfiability problem for $\mathbf{S4} \times \mathbf{S4}$, which turned out to be undecidable [10].

Let us say that a logic L in the language $\mathcal{L}_{\Box_1\Box_2}$ has the *finite topo-product model property* if any non-theorem of L is refuted on a finite product space. Then the logic of products of arbitrary topologies does not have the finite topo-product model property as finite spaces are Alexandroff, and hence validate *com* and *chr*.[1] This remark is not to be confused with the non existence of finite Kripke models: it follows from Theorem 2.7 that every non-theorem of $\mathbf{S4} \oplus \mathbf{S4}$ does indeed fail on a finite model.

6. Adding the true product interior

So far, we have only focused on the horizontal and vertical topologies on the product space, by analogy to products of relational structures. However, the topological semantics suggests a further addition to the language. In this section we investigate the modal logic of products of topological spaces with all three horizontal, vertical, and standard product topologies. We add to the language $\mathcal{L}_{\Box_1\Box_2}$ an extra modal operator $\Box$ with the intended interpretation as the interior operator of the standard product topology.

For two topological spaces $\mathcal{X} = \langle X, \eta\rangle$ and $\mathcal{Y} = \langle Y, \theta\rangle$, we will consider the product $\langle X \times Y, \tau, \tau_1, \tau_2\rangle$ with three topologies, where τ is the standard product topology, τ_1 is the horizontal topology, and τ_2 is the vertical topology. Then $\Box$ is interpreted as follows.

$$(x, y) \models \Box\varphi \quad \text{iff} \quad \exists U \in \eta \text{ and } \exists V \in \theta : U \times V \models \varphi$$

Since $\tau \subseteq \tau_1 \cap \tau_2$, we obtain that the modal principle

$$\Box p \to \Box_1 p \wedge \Box_2 p$$

is valid in product spaces. Our main goal in this section is to show that adding this principle to the fusion of three copies of $\mathbf{S4}$ axiomatizes the logic of products of topological spaces (with three topologies).

Definition 6.1. Let $\mathcal{L}_{\Box,\Box_1,\Box_2}$ be a modal language with three modal operators $\Box$, $\Box_1$, and $\Box_2$. We define the *topological product logic* **TPL** as the least set of formulas in $\mathcal{L}_{\Box,\Box_1,\Box_2}$ containing all axioms of $\mathbf{S4} \oplus \mathbf{S4} \oplus \mathbf{S4}$ plus the axiom $\Box p \to \Box_1 p \wedge \Box_2 p$, and closed under modus ponens, substitution, and $\Box$-, $\Box_1$-, and $\Box_2$-necessitation.

[1]In fact, the same argument implies that no logic in the interval $[\mathbf{S4} \oplus \mathbf{S4}, \mathbf{S4} \times \mathbf{S4}[$ has the finite topo-product model property.

FIGURE 3. $T_{6,2,2}$. The solid lines represent R, the dashed lines represent R_1, and the dotted lines represent R_2. We assume that all dashed and dotted lines are also solid.

Let $T_{6,2,2}$ denote the infinite six branching tree such that each node of $T_{6,2,2}$ is R-related to all six of its immediate successors, R_1-related to the first two, and R_2-related to the last two; R, R_1, and R_2 are taken to be reflexive and transitive. Formally $T_{6,2,2}$ can be defined as $\langle W, R, R_1, R_2\rangle$, where $W = \{0, 1, 2, 3, 4, 5\}^*$,

sRt iff $\exists u \in \{0, 1, 2, 3, 4, 5\}^* : s \cdot u = t$
sR_1t iff $\exists u \in \{0, 1\}^* : s \cdot u = t$
sR_2t iff $\exists u \in \{4, 5\}^* : s \cdot u = t$ (see Figure 3)

Theorem 6.2. **TPL** *is complete with respect to* $T_{6,2,2}$.

Proof. A straightforward generalization of the proofs of Theorem 2.1 and Proposition 2.8. For details see the forthcoming [16]. □

Theorem 6.3. **TPL** *is complete with respect to* $\mathbb{Q} \times \mathbb{Q}$.

Proof. Our strategy is similar to that of the proof of Theorem 5.1. By Theorem 6.2 **TPL** is complete with respect to $T_{6,2,2} = \langle W, R, R_1, R_2\rangle$. We view $T_{6,2,2}$ as equipped with three Alexandroff topologies defined from R, R_1, and R_2. So for completeness of **TPL** with respect to $\mathbb{Q} \times \mathbb{Q}$ it is sufficient to show that there exists a total 3-topo-bisimulation between the $X \times X$ defined in the proof of Theorem 5.1 and $T_{6,2,2}$.

We define h from $X \times X$ onto $T_{6,2,2}$ by recursion following the inductive definition of X (cf. Figure 4(c)): If $(x, y) = (0, 0)$ then we let $h(0, 0)$ be the root r of $T_{6,2,2}$; if $(x, y) \neq (0, 0)$ then there is a unique (u, v) that is labelled before (x, y) such that $(x, y) = (u \pm \frac{1}{3^{n(x,y)-1}}, v)$ or $(x, y) = (u, v \pm \frac{1}{3^{n(x,y)-1}})$ or $(x, y) = (u \pm \frac{1}{3^{n(x,y)-1}}, v \pm \frac{1}{3^{n(x,y)-1}})$. Then we let

FIGURE 4. The first stages of the labelling in the completeness proofs for (a) **S4**, (b) **S4** $\oplus$ **S4**, and (c) **TPL**.

$$h(x,y)=\begin{cases}\text{the left } R_1\text{-successor of } h(u,v) & \text{if } (x,y)=(u-\frac{1}{3^{n(x,y)-1}},v)\\ \text{the right } R_1\text{-successor of } h(u,v) & \text{if } (x,y)=(u+\frac{1}{3^{n(x,y)-1}},v)\\ \text{the left } R_2\text{-successor of } h(u,v) & \text{if } (x,y)=(u,v-\frac{1}{3^{n(x,y)-1}})\\ \text{the right } R_2\text{-successor of } h(u,v) & \text{if } (x,y)=(u,v+\frac{1}{3^{n(x,y)-1}})\\ \text{the first remaining } R\text{-successor} & \text{if } (x,y)=(u+\frac{1}{3^{n(x,y)-1}},v+\frac{1}{3^{n(x,y)-1}})\\ & \text{or } (x,y)=(u-\frac{1}{3^{n(x,y)-1}},v-\frac{1}{3^{n(x,y)-1}})\\ \text{the last remaining } R\text{-successor} & \text{if } (x,y)=(u+\frac{1}{3^{n(x,y)-1}},v-\frac{1}{3^{n(x,y)-1}})\\ & \text{or } (x,y)=(u-\frac{1}{3^{n(x,y)-1}},v+\frac{1}{3^{n(x,y)-1}})\end{cases}$$

Claim 6.4. h is open and continuous with respect to all three topologies.

Proof. The argument that h is open and continuous with respect to τ_1 and τ_2 carries over directly from Claim 5.3. The same technique can be used to show that h is open and continuous with respect to τ. To see this, we observe that

$$\{(x-\frac{1}{3^{n(x,y)}},x+\frac{1}{3^{n(x,y)}})\times(y-\frac{1}{3^{n(x,y)}},y+\frac{1}{3^{n(x,y)}})\ :\ (x,y)\in X\times X\}$$

form a basis for τ on $X\times X$. We also observe that a basis for the topology on $T_{6,2,2}$ defined from R is $\mathcal{B}=\{B_t\}_{t\in T_{6,2,2}}$ where $B_t=\{s\in T_{6,2,2}\ :\ tRs\}$.

To see that h is open, let $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})$ be a basic open for τ. Then the same argument as in Claim 5.3 guarantees that

$$h((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{h(x,y)}.$$

Thus h is open. To see that h is continuous it suffices to show that for each $t \in T_{6,2,2}$, the h-inverse image of B_t belongs to τ. Let $(x, y) \in h^{-1}(B_t)$. Then $tRh(x, y)$. So

$$h((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})) = B_{h(x,y)} \subseteq B_t.$$

Thus there exists an open neighborhood $U = (x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times (y - \frac{1}{3^{n(x,y)}}, y + \frac{1}{3^{n(x,y)}})$ of (x, y) such that $U \subseteq h^{-1}(B_t)$, implying that h is continuous. □

To complete the proof, if **TPL** $\not\vdash \varphi$, then by Theorem 6.2, there is a valuation ν on $T_{6,2,2}$ such that $\langle T_{6,2,2}, \nu\rangle, r \not\models \varphi$. Define a valuation ξ on $X \times X$ by $\xi(p) = h^{-1}(\nu(p))$. Since h is continuous and open with respect to all three topologies and $h(0, 0) = r$, we have that $(0, 0)$ and r are 3-topo-bisimilar. Therefore, $\langle X \times X, \xi\rangle, (0, 0) \not\models \varphi$. Now since $X \times X$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ with respect to all three topologies, it follows that φ is also refutable on $\mathbb{Q} \times \mathbb{Q}$. □

Corollary 6.5. *In the language $\mathcal{L}_{\Box,\Box_1,\Box_2}$,* **TPL** *is the logic of products of arbitrary topologies.*

Incidentally, (using Kripke semantics) it is easy to show that **TPL** is a conservative extension of **S4** $\oplus$ **S4**, and that **S4** $\oplus$ **S4** is a conservative extension of **S4**. Therefore, Theorem 2.4 becomes a corollary of Theorem 5.1, while Theorem 5.1 becomes a corollary of Theorem 6.3.

7. Conclusions and further directions

We introduced the horizontal and vertical topologies on the product of two topological spaces and we showed that the modal logic of products of topological spaces with two horizontal and vertical topologies is the fusion **S4** $\oplus$ **S4**. In addition, we axiomatized the modal logic of products of topological spaces with three horizontal, vertical, and standard product topologies. We conclude by mentioning several open questions that arise naturally from this study.

7.1. **Special spaces.** Although we showed that **S4** $\oplus$ **S4** is complete with respect to $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2\rangle$, and that **TPL** is complete with respect to $\langle \mathbb{Q} \times \mathbb{Q}, \tau, \tau_1, \tau_2\rangle$, it is still an open question what the logics of $\langle \mathbb{R} \times \mathbb{R}, \tau_1, \tau_2\rangle$ and $\langle \mathbb{R} \times \mathbb{R}, \tau, \tau_1, \tau_2\rangle$ are.

Since Alexandroff spaces can be represented as **S4**-frames, it follows from Gabbay at al. [9] that the modal logic of the products of Alexandroff spaces (with horizontal and vertical topologies) is **S4** $\times$ **S4**. On the other hand, it is still unknown what the modal logic is of the products of Alexandroff spaces with arbitrary topological spaces. We conjecture that this logic is **S4** $\oplus$ **S4** $+ com_{\leftarrow} + chr$.

7.2. **Enriching the language.** From a topological perspective, our topological completeness result for **S4** $\oplus$ **S4** seems to suggest that the basic modal language is not expressive enough to model interesting interactions between horizontal and vertical topologies. This suggests to consider richer languages. In adding $\Box$ we have made the first step in this direction, but there are several others that can be taken. For instance, adding the universal modality or nominals.

A very natural extension of the language would be with the common knowledge operator. In the standard Kripke setting, there are several ways of defining common knowledge, but they all turn out to be equivalent (see [3]). In [4] we examine two most prominent such ways and show that in the topological setting the two are in fact distinct. The first defines the common knowledge as an infinite conjunction of claims in the original language, and the second takes common knowledge to be the greatest fixed point of an operator. Thus in our setting the two are:

(1) $C_{1,2}\varphi :=$ an infinite conjunction of all finite nestings of $\Box_1$ and $\Box_2$:

$$\varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_2\varphi \wedge \ldots$$

(2) $K_{1,2}\varphi :=$ the greatest fixed point of the operator $\lambda X.([|\phi|] \cap I_1 X \cap I_2 X)$, as in the following formula of the modal μ-calculus:

$$\nu p.(\varphi \wedge \Box_1 p \wedge \Box_2 p)$$

We argue in [4] that the common knowledge as the greatest fixed point is most interesting from the topological perspective.

7.3. **Further exploration of the connection with Kripke semantics.** We have shown that the topological setting has greater power of discrimination than the relational setting. In particular, topological products validate less principles than products of Kripke frames, and the true product interior modality is not definable in terms of the horizontal and vertical modalities. Several further lines of study might help us understand better the difference between topological products and relational products. Here, we will name two.

Given topological spaces $\langle X, \eta\rangle$ and $\langle Y, \theta\rangle$, call a subset of $X \times Y$ a *block* if it is of the form $A \times B$ for some $A \subseteq X$ and $B \subseteq Y$. Next, call a valuation ν for the product $\langle X \times Y, \tau_1, \tau_2\rangle$ *admissible* if it assigns to each propositional letter a finite union of blocks. Interestingly, when attention is restricted to admissible valuations only, the interaction principles *com* and *chr* become valid again. In fact, we conjecture that the logic obtained in this way (when no restrictions are made on the topological spaces themselves) is precisely $\mathbf{S4} \times \mathbf{S4}$.

The second line of study concerns generalizations of the product construction on Kripke frames. One such generalization is obtained by restricting the universe of admissible product subsets (see, e.g., [5]). The latter is a well-known strategy in relational algebra and arrow logic (see Chapter 7 of [6]). In particular, over such generalized relational products we have that *com* and *chr* are no longer valid, and that the product $\Box$ is no longer definable as $\Box_1\Box_2$.

Incidentally, we believe that the product construction discussed in this paper is of independent interest. Like the fusion and the Kripke product operation, it induces an operation on modal logics. Given normal modal logics L_1, L_2 above $\mathbf{S4}$, we can define $L_1 \times_t L_2$ as the bi-modal logic of the class of products of topological spaces of L_1 and L_2 (with horizontal and vertical topologies). One of our main results, then, tells us that $\mathbf{S4} \times_t \mathbf{S4} = \mathbf{S4} \oplus \mathbf{S4}$. More generally, it is not hard to see that

$$(L_1 \oplus L_2) \ \subseteq\ (L_1 \times_t L_2) \ \subseteq\ (L_1 \times L_2)$$

Many questions arise from this perspective. For instance, does decidability transfer under $\times_t$? This similarity suggests a connection between topological products and generalized relational products.

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