# THE PRESERVATION OF RELEVANCE

DARKO SARENAC AND R. E. JENNINGS

## 1. Introduction

What is it for an implication to be *relevant*? Or to put the matter more directly, when is an antecedent relevant to its consequent? Anderson and Belnap answer these questions by their now well-known variable-sharing requirement. In this paper we try to answer the same questions in a slightly more general fashion drawing heavily on the preservationist approach to implication. The paper falls into three parts. In the first, we present the preservationist approach to implication and other connectives. In the second, we look at a connexivist logic of R.B. Angel and Storrs McCall's, and explain why that logic can be viewed as an historical predecessor of our own preservationist developments[1]. In the final part, we present a series of logics which we claim are in an important sense relevant.

## 2. Preservationist Connectives

### 2.1. Preservationist Implication.

The preservationist approach to implication attempts to understand implication in terms of property-based relations between the antecedent and consequent. Semantically, the approach represents the sentences of the underlying language as *profile vectors*.

**Definition 2.1.** A profile vector is an ordered list of 0's and 1's. Each place in the list represents a property. The leftmost place represents the property $P_1$ and the $i^{th}$ place property $P_i$. 1 in the $i^{th}$ position signifies that the sentence has the property $P_i$ and 0 that the property is absent.

The arity of a profile vector is the number of properties it represents. An *element set*, E, is a collection of distinct profile vectors of the same arity. We let $a, b, c$ range over elements and write $P_i(a) = 1(0)$ to represent the fact that the property $P_i$ is present (absent) in $a$. $D \subseteq E$, is the set of designated elements of E.

$$a \in D \text{ iff } P_1(a) = 1$$

We call $P_1$ the *base property*. $P_2 \ldots P_n$ are called meta-valuational properties. For a number of applications, these are understood as a hierarchy of properties. Rather than being properties of the sentence *per se*, they are taken as properties of previous properties in the list. Thus, $P_4$ is a property of $P_3$ etc. Note, however, that

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[1]See [2], and [9] for the connexivist logic.

not every application demands this hierarchical interpretation. The label remains as a piece of historical reverence.

If $|E| < \omega$, then the connectives of the logical system have finite characteristic matrices. If $|E| = 2^n$, and the arity of the elements is $n$, then $E$ is said to be complete. Otherwise, $E$ is incomplete.

**Example 2.2.** *Let $E = \{11, 01, 00\}$. Then, the arity of the profile vectors is 2. $D = \{11\}$. $E$ is incomplete, since $|E| < 2^n$. $E^+ = E \cup \{10\}$ is the completed element set.*

Any number of relations could obtain between the antecedent and the consequent. For the purposes of this research, we are mostly interested in *quantitative* relations. These relations are used to specify an implication matrix given an element set. The task falls into two parts. The first is to determine pairs of elements a, b, such that, $P_1(a \to b) = 1$, that is, the ones that receive the designated value. The sentence that determines $D$ is called *the preservational profile of implication*.

**Definition 2.3.** A preservational profile of implication is a sentence of the form

$$P_1(\alpha \to \beta) \text{ iff } (P_1(\alpha))\mathbb{R}_{11}(P_1(\beta)) \ldots \& \ldots (P_i(\alpha))\mathbb{R}_{ij}(P_j(\beta)) \ldots \& \ldots (P_n(\alpha))\mathbb{R}_{nn}(P_n(\beta))$$

where $\mathbb{R}_{ij}$ is any of a set of quantitative relations to be described shortly.

The second part of the task is to assign the meta-valuational properties $P_2$ through to $P_n$. As we note elsewhere [see [11]], although the assignment of the meta-valuational properties does not directly determine designation in the implicational matrix, it co-determines the behaviour of the implication in the higher implicational degrees. The meta-valuational properties determine how well behaved the nested implications are. Consequently, the task is important for all but first-degree implicational systems. The sentences which perform this task are collectively called the *non-alethic profile*.

**Definition 2.4.** An *implicational non-alethic* profile is a set of $n-1$ biconditionals $B_j (2 \leq j \leq n)$ where

$$B_j = P_j(\alpha \to \beta) \text{ iff } \phi_j$$

and $\phi_j$ is the set of necessary and sufficient conditions for $P_j(\alpha \to \beta) = 1$.

**Example 2.5.** *Consider the element set $E = \{00, 01, 10, 11\}$. The following preservational profile specifies designation in the implication matrix:*

$$P_1(\alpha \to \beta) = 1 \textit{ iff } P_1(\alpha) \leq P_1(\beta) \textit{ and } P_2(\alpha) \leq P_2(\beta).$$

*The matrix skeleton thus specified would be*

| $\to$ | *00* | *01* | *10* | *11* |
|---|---|---|---|---|
| *00* | *1* | *1* | *1* | *1* |
| *01* | *0* | *1* | *0* | *1* |
| *10* | *0* | *0* | *1* | *1* |
| *11* | *0* | *0* | *0* | *1* |

*If, in addition, we specify the non-alethic profile as, say,*

$$P_2(a \to b) = \begin{cases} 1 & \textit{when } P_2(b) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

*we would obtain the completed implication matrix of the paraconsistent logical system SX* [see [12] for details].

2.1.1. *Quantitative Relations.* As we have hinted in the example above, the basic quantitative relations are $<$, $>$, $=$, $\leq$, $\geq$, min $= 1$, max $= 1$, and *vac*. All the relations up to *vac* are exactly as expected. If, for instance, a preservational profile of an implication requires that $P_i(a) = P_i(b)$, then a necessary condition for the implication's getting a designated value is that $i^{th}$ property be assigned identically in both the antecedent and the consequent. The special relation *vac* [for vacuous] holding for the $i^{th}$ property specifies that the $i^{th}$ property sets no further restriction. An implication the preservational profile of which consisted of this special relation for every property would always be designated. To avoid clutter, we omit any pair for which this relation holds, that is, any pair such that $P_i(a) \, vac \, P_j(b)$.

The quantitative relations are variably used in a preservational profile. In the above example, the same relation ($\leq$) holds across the property range, that is, the implication preserves both $P_1$ and $P_2$. This is not always the case, however. Some preservational profiles have some relation holding for some property, and a different relation for a different property. We will see an example shortly. Classical propositional logic [henceforth PL] will generate some examples. In the case of PL, it should be noted, the elements are unary profile vectors.

**Example 2.6.** *Suppose we set the following preservational requirement to an implicational connective:*

$$P_1(\alpha \to_< \beta) \textit{ iff } P_1(\alpha) < P_1(\beta).$$

*We obtain an implicational connective for which* $p \to_< p$ *is not a theorem. This connective is the classical truth-function* $\neg(q \to_{PL} p)$.

*Suppose, now, we weaken the preservationist requirements and allow:*

$$P_1(\alpha \to_\supset \beta) \textit{ iff } P_1(\alpha) \leq P_1(\beta).$$

*Then, we get* $p \to_\supset p$ *back. In fact, assuming the standard PL negation, we get the material conditional of PL with exactly all of its tautologies.*

*For the final example, suppose we set the requirements to:*

$$P_1(\alpha \to_\equiv \beta) \textit{ iff } P_1(\alpha) = P_1(\beta).$$

*In this case, the implication is the thinly disguised classical equivalence.*

In the unary case, all the connectives are variations on the classical theme. Things become more interesting only as we add more properties. Now we can not only have different properties with different relations holding between the antecedent and the consequent, but we can also place relational restrictions on only some of the properties. The quantitative relation *vac* enables this move. Thus, we can have a profile on a complete element set with 5 properties with the requirement,

$$P_1(\alpha \rightarrow \beta) = 1 \text{ iff } P_1(\alpha) \textit{ vac } P_1(\beta), \text{ and } P_2(\alpha) \leq P_2(\beta), \text{ and}$$
$$P_3(\alpha) \leq P_3(\beta), \text{ and } P_4(\alpha) \textit{ vac } P_4(\beta), \text{ and } P_5(\alpha) \textit{ vac } P_5(\beta).$$

The latter possibility is particularly interesting in that it can be used to characterize modalized implications (strict implication, entailment etc.) of some of the Kripke systems weaker than S5[2]. The last, but by no means the least interesting, option is to have distinct properties as the relata of a quantitative relation. For example, an implication connective can be defined over a three-property complete element set as follows,

$$P_1(\alpha \rightarrow \beta) = 1 \text{ iff } P_1(\alpha) \leq P_1(\beta) \text{ and } (P_1(\alpha) \leq P_2(\beta) \text{ or } P_1(\alpha) \leq P_3(\beta)).$$

Finally, we call an implicational connective *normal* if it satisfies

$$P_1(\beta) < P_1(\alpha) \Rightarrow P_1(\alpha \rightarrow \beta) = 0.$$

Every implication we consider here is normal.

2.2. **Preservationist Negation.** As we have mentioned, the matrix elements as understood in this line of research are lists of binary properties. What this means is that any non-vacuous unary operator must reverse some of the properties for some of the cases. We define the negation operator in that spirit.

**Definition 2.7.** Negation is any unary operator such that

$$\exists P_i, \langle 1 \leq i \leq n \rangle, P_i(\neg\alpha) = |P_i(\alpha) - 1|.$$

In other words, a negation reverses some of the properties in the range $P_1$–$P_n$.

We call a negation operator *normal* if it reverses $P_1$ and *classical* if it is normal and reverses $P_1$ only. A part of the reason for the latter piece of terminology is inductive. In the past, our research strategies have made use of an internal negation which reverses $P_1$ alone. This negation not only behaved classically with respect to the conjunction and disjunction we devised for our preservationist purposes, but it also succeeded in converting Heyting's intuitionist disjunction into its tamed classical counterpart [see [7], [11], and [12]]. More importantly, however, the designative property signifies the nostalgic yearning for the security of our classical upbringing, and that property, we believe, should behave entirely classically. As we argue elsewhere [see [11], and [12] for details], it is the complexity of the domains of application that makes PL unsatisfactory, not some internal flaw. When the domain is as simple as the polarities of a binary property, PL works just fine.

Note that our negation does not accommodate the kinds of negation that are constant for some of the values. For instance, the negation in Priest's and Routley's Logic of Paradox would not fall under our concept of negation. Such negations could easily be accommodated, however, if we weakened the requirement to reversal of some of the properties some of the time. We work with such a concept of negation in [12] and [13]. For the purposes of this project, there is no need to do so.

[2]If the points of the universe are understood as binary properties, actual world as the property which determines designation ($P_1$), and the set of relata of the actual world as the subset of properties to be preserved.

2.3. **Preservationist Conjunction and Disjunction.** Our treatment of conjunction and disjunction represents a very slight modification of the treatment of implication. Like implication, these connectives are specified through a preservationist profile together with a non-alethic one. These are modified in obvious ways, and we leave to the reader to see what these obvious modifications are. A conjunction is *normal* if $P_1(a \wedge b) \leq \min(P_1(a), P_1(b))$. A disjunction is *normal* if $P_1(a \vee b) \leq \max(P_1(a), P_1(b))$. If $P_1(a \wedge b) = \min(P_1(a), P_1(b))$, then a conjunction is *classical*. If a disjunction satisfies $P_1(a \vee b) = \max(P_1(a), P_1(b))$, then it is classical. Note that we only give sufficiency for the classicality of negation, disjunction, and conjunction. Whether a connective is classical or not largely depends on how it interacts with other standard connectives. We give a sufficiency condition because the conditions above ensure classical interactions among negation, conjunction, and disjunction. This is not to say, however, that connectives with different profiles would not behave classically.

**Example 2.8.** *Let us consider an example of a non-classical conjunction. Let our element set, $E$, be a complete collection of binary profile vectors. Let the preservationist profile for conjunction be*

$$P_1(a \wedge b) = 1 \textit{ iff } \min(P_1(a), P_1(b)) = 1 \textit{ and } P_2(a) = P_2(b).$$

*This enables us to construct the designative portion of the conjunction matrix.*

| $\wedge$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 0 |
| 01 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 1 | 0 |
| 11 | 0 | 0 | 0 | 1 |

*We add a non-alethic profile.*

$$P_2(a \wedge b) = 1 \textit{ iff } P_2(a) = P_2(b) = 1.$$

*Then the completed matrix is*

| $\wedge$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 00 | 00 | 00 | 00 |
| 01 | 00 | 01 | 00 | 01 |
| 10 | 00 | 00 | 10 | 00 |
| 11 | 00 | 01 | 00 | 11 |

*The first thing to note that the conjunction is normal. The value of the designative property under the conjunction is never greater than the min of the values of the designative property of the elements. The conjunction, however, is nothing like classical. The conjunction could be false (non-designated) although both conjuncts are true (designated). If we attempt to define the disjunction in the standard fashion as $\neg(\neg p \wedge \neg q)$, things become even wilder.*

| $\vee$ | *00* | *01* | *10* | *11* |
|---|---|---|---|---|
| *00* | *00* | *10* | *10* | *10* |
| *01* | *10* | *01* | *10* | *11* |
| *10* | *10* | *10* | *10* | *10* |
| *11* | *10* | *11* | *10* | *11* |

This disjunction is neither classical nor normal. The disjunction could be true despite the fact that neither of its disjuncts is. If designation signifies something like weak necessity, say, this, of course, comes as no surprise. In the ternary logics of Jennings and Schotch, for example, $\Box p$ and $\Box q$ could hold without $\Box(p \wedge q)$ holding. In addition, $\Box(p \vee q)$ could hold without either $\Box p$ or $\Box q$ [see [14] and [15]]. There is also a relevantist interpretation for the conjunction as we will see in the sequel.

## 3. Connexivism and Relevance

In [2], R.B. Angel presents a set of matrices with an implicational connective which he claims captures the notion of *subjunctive conditional*. Then, in [9], Storrs McCall takes over the matrices arguing that, instead, they are suited for a connexivist logic. Here are the matrices:

| $\rightarrow$ | 1 | 2 | 3 | 4 | $\neg$ |
|---|---|---|---|---|---|
| 1 | 1 | 4 | 3 | 4 | 4 |
| 2 | 4 | 1 | 4 | 3 | 3 |
| 3 | 1 | 4 | 1 | 4 | 2 |
| 4 | 4 | 1 | 4 | 1 | 1 |

| $\wedge$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 3 | 3 | 4 |
| 2 | 1 | 2 | 4 | 3 |
| 3 | 3 | 3 | 3 | 4 |
| 4 | 4 | 3 | 4 | 3 |

It is easy to see that both distinctive connexivist theses—Aristotle, $\neg(\neg p \rightarrow p)$, and Boethius, $(p \rightarrow q) \rightarrow \neg(p \rightarrow \neg q)$—hold. As McCall emphasizes, the system has numerous other virtuous properties. It avoids various 'fallacies of necessity' as well as some 'fallacies of relevance' [see [9] and especially [1] for the discussion of the fallacies]. That there is a connection between McCall's connexive system and relevance is a fact known from the inception of the system. Anderson and Belnap include CC1, as McCall calls the system, in the list of systems related to relevant logics in [1]. What we explore here is what we take to be the basis of relevance in CC1. But first, we need to translate the matrices into a preservationist language. The following translation of the elements of CC1 into binary profile vectors will do the job. Let $x$ range over elements of CC1, and let $f$ be a translation function satisfying

$$P_1(f(x)) = 1 \text{ iff } x \leq 2, \text{ and } P_2(f(x)) = 1 \text{ iff } x \text{ is odd.}$$

Then, the binary property-based matrices are

| $\rightarrow$ | 11 | 10 | 01 | 00 | $\neg$ |
|---|---|---|---|---|---|
| 11 | 11 | 00 | 01 | 00 | 00 |
| 10 | 00 | 11 | 00 | 01 | 01 |
| 01 | 11 | 00 | 11 | 00 | 10 |
| 00 | 00 | 11 | 00 | 11 | 11 |

| $\wedge$ | 11 | 10 | 01 | 00 |
|---|---|---|---|---|
| 11 | 11 | 10 | 00 | 01 |
| 10 | 11 | 10 | 00 | 01 |
| 01 | 01 | 01 | 01 | 00 |
| 00 | 00 | 10 | 00 | 01 |

All three connectives are normal, and the conjunction is classical. In addition, the conjunction-negation fragment is characteristic for PL. It is easy to check that the two connectives define $\supset$, $\vee$, and $\equiv$ in the standard fashion, and those validate your choice of a preferred PL axiomatization. Interesting formulae of CC1 involve negations and arrows. Under our translation, the negation reverses both properties in binary profile vectors of the elements. Thus, the negation of 11 is 00. This fact poses some difficulty to an attempt to read $P_1$ as truth, and $P_2$ as necessitation. On this reading, a negation of a necessarily true proposition is contingently false, and a negation of a necessary falsehood is a contingent truth. This way of approaching negation in modal contexts is rare. It is a cross of two more standard approaches, internalist and externalist. In the latter approach, a negation of a necessary proposition $p$ is *it is not necessary that* $p$, whereas in the former, it is *necessarily not* $p$.

The implicational connective is by far the most inspiring. To facilitate its analysis, we dissect it into the preservational and the non-alethic profile.

3.1. **Implication of CC1.** The preservational profile of the implication is

$$P_1(\alpha \to \beta) = 1 \text{ iff } P_1(\alpha) \leq P_1(\beta) \text{ and } P_2(\alpha) = P_2(\beta).$$

The standard set by $P_1$ is the usual $\leq$-relation. The additional requirement set by $P_2$ is that the antecedent and the consequent share 'an equal amount of $P_2$,' or less metaphorically, that $P_2(a) = P_2(b)$. Let us put this significant point yet another way. If the implicational connective recieves a designated value, then either both the antecedent and the consequent have property $P_2$ or they both lack that property.

The non-alethic profile is a singleton, where $\phi_2$ is

$$P_2(\alpha) = P_2(\beta).$$

In [9], McCall acknowledges that propositional equivalence is the simplest but least attractive connexivist implication. CC1 is intended to answer affirmatively the question whether there are connexivist implications for which $p \to q$ does not entail $q \to p$. The preservationist rendering of this logic reveals that the connexivism of this implication is achieved by combining the properties of the classical material implication with the properties of the classical equivalence. When, in addition, the negation is the strong connective described above, the ingredient list is complete. It is easy to see that the weaker classical negation would not be appropriate for connexivist purposes. Both theses, Aristotle and Boethius, fail when the strong connexivist negation is replaced with the weaker classical one.

Eventually, we will replace McCall's negation with the classical one, but before we move to more technical details, let us briefly consider McCall's implication under the heading of *relevance*. As we promised, our aim is to interpret this implication as a *relevant* implication. One very general way of answering the question of when a sentence $\alpha$ is relevant to sentence $\beta$ is by insisting that relevance requires property sharing. Thus, $\alpha$ and $\beta$ are relevant to each other if they share some properties, and the more properties they share the more mutually relevant they are. Applying this reasoning to implication, we say that the antecedent $\alpha$ is relevant to the consequent $\beta$ if the two share some specified set of properties. Rendered formally, $\exists i$, $P_i(\alpha) = P_i(\beta)$. Now consider a normal implication over the complete set of binary profile vectors. There is only one meta-valuational property, and if for relevance we require that the antecedent and the consequent share that property, we obtain exactly the preservational profile of McCall's implicational connective. This profile of implication provides a seed for a series of relevant logics to which we now turn.

## 4. Preserving Relevance

Our relevant explorations focus on implication and conjunction. Throughout, we will insist on classical negation, and, to simplify some of the meta-theory, we introduce classical disjunction. The two latter connectives, in turn, define all other PL connectives in the standard way. We call the logics in the sequence $PR_i$, where $PR$ stands for property relevance, and $i$ is the number of meta-valuational properties. The matrices of the two relevant connectives—conjunction and implication—of $PR_1$ are

| $\circ$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 00 | 00 | 00 | 00 |
| 01 | 00 | 01 | 00 | 01 |
| 10 | 00 | 00 | 10 | 00 |
| 11 | 00 | 01 | 00 | 11 |

| $\rightarrow$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 10 | 01 | 10 | 01 |
| 01 | 00 | 11 | 00 | 11 |
| 10 | 00 | 01 | 10 | 01 |
| 11 | 00 | 01 | 00 | 11 |

Both connectives are normal and neither is classical. The conjunction matrix is identical to the matrix described in the example 2.8. Hence, the preservational and the non-alethic profile carry over. The idea behind this connective is that for a relevant conjunction to receive a designated value it is not sufficient that both conjuncts be designated. It is necessary, in addition, that the two conjuncts be relevant to each other. In other words, they have to share the meta-valuational properties. A similar requirement could be set to a disjunction connective. As we point out above, such connective cannot be defined as $\neg(\neg p \circ \neg q)$.

The preservational profile for implication is that of CC1. The non-alethic profile, however, differs. As Routley and Montgomery point out in [10], McCall's implication misbehaves when nested, and when iterated conjunctions or disjunctions are used as antecedents and consequent. Some of the misbehavior is corrected by the following non-alethic profile

$$P_2(\alpha \rightarrow \beta) = P_2(\beta).$$

In addition, this profile is much easier to justify from the relevant point of view. The conditional receives the properties of the consequent, thus ensuring that the nested antecedents are actually relevant to the consequent. This strategy, as we point out in [12], also ensures paraconsistency of the conditional fragment of the logic. The remaining connectives are

| $\alpha$ | $\neg\alpha$ |
|---|---|
| 00 | 10 |
| 01 | 11 |
| 10 | 00 |
| 11 | 01 |

| $\vee$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 00 | 00 | 10 | 11 |
| 01 | 00 | 01 | 10 | 11 |
| 10 | 10 | 10 | 10 | 11 |
| 11 | 11 | 11 | 11 | 11 |

together with an appropriate assortment of undesignated constants. In the case of $PR_1$ the constant $\perp = 01$ suffices. The constants are used in the semantics-to-syntax translation. The translation, to be described shortly, amounts to soundness and completeness of any finite logic in the series. Thus, $\perp \rightarrow \alpha$ is used to represent syntactically the semantic fact that $P_2(\alpha) = 1$. For every new property added to the semantics, a false constant representing that property is added to the syntax. For instance, if the arity of the profile vectors is 4, we add the *falsa* $-1$, $-2$, and $-3$ representing respectivelly 0100, 0010, and 0001. For some property $P_i$, let $F_{P_i} = \{\perp_j \mid P_i(\perp_j) = 1\}$. It will turn out that the syntactic representative for $P_i(\alpha) = 1$ is $\bigvee(\beta \to \alpha)$, $\beta \in F_{P_i}$.

4.1. **PR series.** $PR_n$ is obtained from $PR_{n-1}$ by adding an additional property to the profile vectors of $PR_{n-1}$. The matrices for $\to$, and $\circ$ are obtained through the following preservationist and non-alethic profiles:
$(\to)$
$P_1(\alpha \to \beta) = 1$ iff $P_1(\alpha) \leq P_1(\beta)$ and $\forall i(2 \leq i \leq n) P_i(\alpha) = P_i(\beta)$.
$\forall i(2 \leq i \leq n)(P_i(\alpha \to \beta) = P_i(\beta))$.
$(\circ)$
$P_1(a \wedge b) = 1$ iff $min(P_1(a), P_1(b)) = 1$ and $\forall i(2 \leq i \leq n) P_i(a) = P_i(b)$.
$\forall i(2 \leq i \leq n)(P_i(a \wedge b) = 1$ iff $P_i(a) = P_i(b) = 1)$.

In every degree, the disjunction and negation are classical.

$$\forall i(2 \leq i \leq n)(P_i(\neg a) = P_i(a)),$$

and

$$\begin{aligned} \forall i(2 \leq i \leq n) P_i(a \vee b) &= \min(P_i(a),\, P_i(b)) \text{ if } a, b \notin D, \\ &= \max(P_i(a),\, P_i(b)) \text{ if } a, b \in D \\ &= P_i(a) \text{ if } a \in D \text{ and } b \text{ is not, and } \textit{vice versa}. \end{aligned}$$

Each new property added represents a domain of discourse. A sentence is relevant to another sentence if they share a domain of discourse. In fact, on this approach, the sentences have to share *all* domains of discourse. A next research stage would, perhaps, be to require only some specific domains to be shared by weakening the universal quantifier in the preservational profiles for $\to$ and $\circ$ . Or perhaps to order domains by significance. At any rate, the more properties, the more domains. And, the more domains, the more classes of mutually irrelevant sentences.

4.2. **Generalized Completeness for PR Systems.** We approach completeness from a literalist point of view. Every semantic property has its syntactic representative in the guise of the set of the above mentioned falsa. As mentioned above, $\bigvee(\beta \to \alpha)$, $\beta \in F_{P_i}$ represents the fact that $P_i(\alpha) = 1$. $\neg(\bigvee(\beta \to \alpha)$, $\beta \in F_{P_i})$ represents the fact that the property is absent. At this point, the reason for making the full force of PL available becomes apparent. Having the force of PL, we can now translate the preservational and the non-alethic profiles directly into axioms of the logic. Those axioms are obviously sound and sufficient for completeness. Using the standard PL connectives,

$$\begin{aligned} p \wedge q &=_{\text{Def.}} \neg(\neg p \vee \neg q) \\ p \supset q &=_{\text{Def.}} \neg p \vee q \\ p \equiv q &=_{\text{Def.}} (p \supset q) \wedge (q \supset p) \end{aligned}$$

we proceed with the translation as follows. Calling the translation function $\mathbb{T}$,

(i) $\mathbb{T}(P_1(\alpha) = 1) = \alpha$
(ii) $\forall i(2 \leq i \leq n)\mathbb{T}(P_i(\alpha) = 1) = (\bigvee(\beta \to \alpha),\, \beta \in F_{P_i}$
(iii) $\forall i, \mathbb{T}(P_i(\alpha) = 0) = \neg(\mathbb{T}(P_i(\alpha) = 1))$
(vi) $\mathbb{T}(P_i(\alpha) \leq P_j(\beta)) = \mathbb{T}(P_i(\alpha)) \supset \mathbb{T}(P_j(\beta))$
(v) $\mathbb{T}(P_i(\alpha) = P_j(\beta)) = \mathbb{T}(P_i(\alpha)) \equiv \mathbb{T}(P_j(\beta))$
(vi) $\mathbb{T}(P_i(\alpha) \geq P_j(\beta)) = \neg(\mathbb{T}(P_i(\alpha))) \supset \neg(\mathbb{T}(P_j(\beta)))$
(vii) $\mathbb{T}(P_i(\alpha) < P_j(\beta)) = \neg(\neg(\mathbb{T}(P_i(\alpha))) \supset \neg(\mathbb{T}(P_j(\beta))))$
(viii) $\mathbb{T}(P_i(\alpha) > P_j(\beta)) = \neg((\mathbb{T}(P_i(\alpha))) \supset (\mathbb{T}(P_j(\beta))))$
(ix) $\mathbb{T}(\min(P_i(\alpha), P_j(\beta)) = 1) = \mathbb{T}(P_i(\alpha)) \wedge \mathbb{T}(P_j(\beta))$
(x) $\mathbb{T}(\max(P_i(\alpha), P_j(\beta)) = 1) = \mathbb{T}(P_i(\alpha)) \vee \mathbb{T}(P_j(\beta))$
(xi) $\mathbb{T}((P_i(\alpha)\, vac\, P_j(\beta)) = 1) = \neg\bot_n$

Let us look at an example of the application of $\mathbb{T}$. Applying $\mathbb{T}$ to the preservational profile of implication we get,

$$(p \to q) \equiv ((p \supset q) \wedge ((\bot \to p) \equiv (\bot \to p))).$$

When, in addition, we translate the non-alethic profile into

$$(\bot \to (p \to q)) \equiv (\bot \to q)$$

we have all the implicational axioms needed for the completeness proof.

The completeness proof itself is a fairly standard Henkin-style construction. Maximal $PR_i$-consistent sets are almost entirely standard. The Henkin-model is an $n+1$-tuple consisting of the maximal $PR_i$-consistent set and the $n$ properties which comprise the profile vectors. The fundamental theorem splits into $n$ stages, a stage for each property. We show that

$$\alpha \in \Sigma \text{ iff } P_1^{\Sigma}(\alpha) = 1,$$

and that

$$\forall (2 \leq i \leq n)(\bigvee(\beta \to \alpha),\, \beta \in F_{P_i}) \in \Sigma \text{ iff } P_i^{\Sigma}(\alpha) = 1,$$

where $\Sigma$ is a $PR_n$ maximal consistent set. The translation function $\mathbb{T}$ ensures that each of the $n$ stages of the proof for each of the connectives proceeds as smoothly as possible.

4.3. **A Countable Case.** There is a PR-system in which the intuitions of the preservationist approach to relevance seem to converge with the Anderson-Belnap variable-sharing requirement. Consider a set $\mathbb{P}$ of countably many properties $P_i$, and a countable set V of indexed propositional variables $p_i$. We concentrate our attention on the class of assignments satisfying the following requirement

$$\forall p_i \in V, \exists P_j \in \mathbb{P},\, P_j(p_i) = 1, \text{ and } \forall p_k \in V \neq p_i, P_j(p_k) = 0.$$

In other words, the class of assignments under which each propositional variable gets assigned a unique property.

The preservational and the non-alethic profiles for $\to$ easily generalize to the countable case. It is obvious that if the consequent has a property uniquely associated with some variable $p_i$, the implicational sentence will receive a designated value only if the antecedent contains the variable. This immediately reveals the strength of the requirement, however. Any non-designative property present in the consequent will have to be present in the antecedent. If these properties are uniquely associated with propositional variables, then the variable-sharing requirement will be rather strong.

Two ways of weakening the preservationist profile of implication immediately come to mind. The first is to replace = with $\leq$. The preservational profile then becomes

$$P_1(\alpha \to \beta) = 1 \text{ iff } \forall i(1 \leq i \leq \omega)P_i(\alpha) \leq P_i(\beta).$$

The second is to require only some of the properties to be shared. Thus,

$$P_1(\alpha \to \beta) = 1 \text{ iff } P_1(\alpha) \leq P_1(\beta) \text{ and } \exists i(2 \leq i \leq \omega)P_i(\alpha) = P_i(\beta) = 1.$$

Each of the three preservationist implication satisfies the variable-sharing requirement. An interesting open question is which, if any, aligns with Anderson-Belnap requirement.

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Laboratory for Logic and Experimental Philosophy, Simon Fraser University, Burnaby, British Columbia, Canada

*E-mail address*: jennings@sfu.ca and dsarenac@sfu.ca

*URL*: http://www.sfu.ca/llep