PARACONSISTENCY, IMPLICATION, AND TRUTH

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1. Paraconsistency and Truth

The mainstream of paraconsistent research is divided into two opposing subdivisions, one of them, the dialetheist and another that we shall call the confusionist. The first group, mainly associated with the Australian continent, maintains that there is at least one sentence, α, such that both α and $\neg\alpha$ are true. The paradox of the liar and related paradoxes are the most commonly cited examples. The incentive, then, for devising a paraconsistent logic is the fact that the classical logic (henceforth CL) mistreats some of the true sentences. CL endorses the rule of *conjunction introduction* which together with the rule *Ex contradictione quodlibet*—everything follows from a contradiction—enables us to prove any sentence from the above described pair. Put forcefully, CL endorses a slogan 'if a contradiction is true, anything is', which in the presence of a true contradiction, leads into obvious trouble.

The second group, usually seen as more conservative, maintains the view that no contradictory pair of sentences could ever be true. The usual position is that somehow or other we know that *a priori*. So, if we had direct access to true sentences, CL would always serve us well. Despite that, we still need a paraconsistent logic. The breakdown occurs as the result of our cognitive limitations. We, our databases, and other items of logical and cognitive interest do not have direct access to truth, but have to rely on often unreliable processing channels. Nuel Belnap in [1] gives an example of a database with several channels of input. If one channel inputs α and a different channel inputs $\neg\alpha$ a classical database is in trouble. To sum up, we often have to deal with contradictions, but if a contradiction 'happens', it is a result of a confusion on some level of information processing.

Suffice it to say, of the two firms, the confusionist one has attracted the more customers. The Aristotelean *law of non-contradiction* is so deeply entrenched in the philosophical tradition that only the bravest of recusants have dared to dissent. We admire these brave souls, even though, as we suppose, the moves they make are based on erroneous convictions. And both views, in our opinion, are so based.

1.1. Do we understand truth?

The culprit is the concept of truth. Truth and preservation of truth are taken to be of core interest to logicians. It is often said that the preservation of truth in all situations is the aim of the 'true' logical system. We, here, propose a slightly different view. The concept of truth is an intuitive, folk-theoretic, and poorly understood notion. As such, it is an unsuitable base for

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anything but a very crude theoretical endeavour. In addition, no one has defined the notion for serious scientific use, and physical science manages quite well without any definition whatever. On the one hand, these facts make reliance upon truth-conditions and truth-preservation suspect as the sole bases for semantical studies; on the other, they place the well-defined but purely formal notions of valuation and designated value at the disposal of interpretational market forces, where they properly belong. It is all very well to *read* 1 and 0 as truth-values, but we deceive ourselves if we suppose that that reading constitutes an interpretation.

It should be noted that, although these remarks on truth will seem fairly radical from a philosophical standpoint, their actual practical consequences are fairly minor. We are not proposing linguistic revisionism: the everyday usage of the concept does not suffer. But, the domain of everyday linguistic interactions is where the concept belongs. As for mathematics and the physical sciences, they devise their own internal criteria for determining which sentences are to be incorporated into the account and which rejected as false. As a part of the language, a scientist works with sentences which for various reasons were incorporated into her account. But there is no reason to endow the scientist with sophisticated views on the nature of truth. She might very well suppose that her theories are true, and for that, she needs no deep understanding of truth. It is just that in her laboratory confirmation, expressibility etc. are as much as she can justifiably have. A mathematician incorporates a sentence into the account if it can be derived from previously accepted sentences by the application of a set of predefined rules, and, so on. Take away truth, and nothing is lost (save perhaps some of the rhetorical impact and inspired visions of realist metaphysics).

In fact, something is gained, namely the freedom from the burdensome folk-theoretic intuitions about truth. Our only limitation is the expressive power of our formalism. For instance, now we can designate 0 in the binary-valued semantics obtaining an interesting dual of CL. The dual enables us to preserve falsity when such is needed.

On this account, the question of whether a formal system is virtuous is not decided by whether it preserves truth across all admissible situations, but by practical considerations. A formal system is a contrivance. Like any other contrivance, its virtue is defined in relation to an application. The sociology of logical research seems to confirm this view. What, we may ask, would be more highly valued, a formal system that preserves truth across all admissible situations, or a formal system that helps solve P=?NP? Would anyone fund US$ one million for the former? Someone has for the latter.

Instead of trying to capture the elusive intuitions on truth, admittedly a hard and likely impossible enterprise, we settle for a very modest endeavour. We define a formal setting and a class of measures which distinguish classes of system in the setting. The measures define properties of classes of sentences or relations among the classes of sentences. An immediate virtue of the approach is that is it moves logical disputes from the realm of battling intuitions into the realm of well-defined objects and properties.

As an instance, such an approach helps dissolve disputes akin to the one between dialetheist and confusionist. The question of whether there are true (read 'designated') contradictions is now easy to settle. First of all, it must be answered in relation to a formal system. If the question, for example, is whether there are

CL-true contradictions, the answer is plainly 'no'. CL assigns truth-values to propositional variables, and the truth-values of the rest of the sentences are a function of that assignment. And for every α, $\alpha \wedge \neg\alpha$ is assigned an undesignated value. Without relying on murky intuitions, we can now ask not only whether a contradiction can have a designated value relative to a system, but also whether and to what degree contradictions are inferentially tolerated by the formal system. Once again, the degree of tolerance is the degree to which the system satisfies some well-defined formal measure.

This suggests a way of settling another muddled issue in paraconsistent research: the issue as to when a formal system is paraconsistent. First of all, the question as it stands is too general, and should more properly be particularized to sublinguistic features of the system. A logic is paraconsistent relative to some specific set ($\{\bot\}$ or $\{\alpha, \neg\alpha\}$), some connective ($\rightarrow$), the entailment relation, or some other significant feature of the system when it satisfies some well-defined measure. The measure describes the behaviour of an aspect of the logic. Whether a system is to be called paraconsistent when it satisfies some specific measure becomes simply a terminological dispute.

In this spirit, we could designate a set of standards, such as

$$\exists\alpha, \exists\beta, \{\alpha, \neg\alpha\} \nvdash \beta,$$
$$\forall\alpha, \exists\beta, \{\alpha, \neg\alpha\} \nvdash \beta,$$
$$\exists\alpha, \forall\beta, \{\alpha, \neg\alpha\} \nvdash \beta, \text{ and even}$$
$$\forall\alpha, \forall\beta, \{\alpha, \neg\alpha\} \nvdash \beta$$

and measure various systems against them. The systems that survive should not be thought of as the systems that *live up to* the standards of some or other measure, rather, they should be thought of as those that satisfy minimal standards for some or other application (practical or not). The measures simply put into relief some significant properties of a system. The idealized situation would be as follows. A domain of application of the logic dictates certain restrictions and positive requirements of the formalism. The first move of a researcher is to translate the set of requirements and restrictions (such as preservation of truth, non-triviality over contradiction, preservation of normality of a function) into a set of well-defined and purely formal measures. Then, having verified that the measures more or less match the informal understanding of the domain, all that is left to the researcher is to find a formal system that best satisfies the required measures. Of course neither of the steps is as easy as we make it sound. Trying to capture formally the limitations of the domain is likely to be the hardest. If there is disagreement among logicians, it is often over what the domain of logic should be and whether the peculiarities of the domain have been captured correctly. But once we move into the formal realm, there is much less room for disagreement. If the system satisfies the measures, it is worth consideration.

All the above-listed standards pertain to the $\vdash$-relation. Our goal in this paper is to present a measure which does not directly deal with the $\vdash$-relation. The measure will deal with implicational behaviour relative to a set. It will assign a number to an inconsistent set according to how difficult it is implicationally to induce an explosion from that set. Before we introduce the measure, we need to spell out some of the details of our general approach to logic.

2. The Preservationist approach

The core slogan of the preservationist approach to logic is 'find desirable properties of sentences or sets of sentences and preserve them'. Two sublinguistic features—the entailment relation ($\vDash$) and the implication function ($\rightarrow$)—have so far carried most of the preservationist burden.

Definition 2.1. If $\vDash$ preserves a property P, and the relation holds between Σ and α, then if the set Σ has the property P, α has the property as well.

For example, the classical logic preserves consistency, since Σ is consistent if and only if $Cn_{CL}(\Sigma)$ is [for details, see [3]]. Various preservationist restrictions on $\vDash$ have been examined in some detail in [2], [3], [6], [10], and [11].

Definition 2.2. The connective $\rightarrow$ preserves properties of sentences. If $\rightarrow$ preserves P, and $\alpha \rightarrow \beta$ receives a designated value, then if α has P, β has P.

The present study is a sequel to [5] and [7]. We further explore preservational properties of a class of implicational connectives. The implicational approach to preservation examines what happens when the implicational connective is required to preserve various properties. We focus on a particular class of properties, called *meta-valuational properties*, and the class of implicational connectives required to preserve various dimensions of these properties. Meta-valuational properties form a hierarchy of properties. The hierarchy begins with a basic property which behaves like the standard valuation of classical propositional logic. We begin by generalizing the preservationist approach to implication from [5] and [7] to an arbitrary binary connective and then evaluate a class of implicational connectives according to a criterion that we label the *Nobel measure*.

3. Preservationist Connectives

3.1. **Atomic Sentences.** Our approach is semantic. First, we describe the semantics of the atomic sentences. Atomic sentences, on this approach, are variables ranging over a set, E, the set of semantic elements. Let a, b, and c range over the elements of E. The set of properties $\mathbb{P}_E$ is the set of binary properties distinguished by the elements of E. $P_i \in \mathbb{P}_E$ is represented as a function $P_i : E \rightarrow 2$. Elements of E are called *profile vectors*.

Definition 3.1. A *profile vector* is an ordered list of 0's and 1's. Each place in the list represents a property. The leftmost place represents the property P_1 and the i^{th} place, property P_i. 1 in the i^{th} position signifies that the sentence has the property P_i and 0 that the property is absent.

The *arity* of a profile vector is $|\mathbb{P}_E|$. The set E, then, is a collection of distinct profile vectors of the same arity. We write $P_i(a) = 1(0)$ to represent the fact that the property P_i is present (absent) in a. $D \subseteq E$, is the set of designated elements of E.

$$a \in D \text{ iff } P_1(a) = 1$$

We call P_1 the *root property*. $P_2 \ldots P_n$ are called meta-valuational properties. For the purposes of the current application, these are understood as a hierarchy of properties. Rather than being understood as properties of the sentence *per se*, they are taken as properties of the previous properties in the list. Thus, P_4 is a property of P_3 etc. [for details see [7]].

If $|E| < \omega$, then the connectives of the logical system have finite characteristic matrices. If $|E| = 2^n$ where the arity of the elements is n, then E is said to be *complete*. Otherwise, E is *incomplete*.

Example 3.2. *Let* $\mathbb{P}_E = \{P_1, P_2, P_3\}$*, and* $E = \{111, 101, 100, 011, 010, 000\}$. $D = \{111, 101, 100\}$*. The arity of the profile vectors is 3. E is incomplete, since* $|E| < 2^3$*. Adding* 110 *and* 001 *to E extends E to a complete element set,* E^+.

3.2. **Binary Connectives.** A binary connective $*$ for $|\mathbb{P}_E| = n$ is a collection of n functions $P_1^* \dots P_n^*, P_i^* : E \times E \to 2$. P_i^* is the extension function for P_i, and is defined as follows

$$P_i^*(a * b) = 1 \text{ iff } (f_{11}((P_1(a)), (P_1(\beta))) = 1, \dots \& \\ \dots f_{ij}((P_i(a)), (P_j(\beta))) = 1, \dots \& \dots f_{nn}((P_n(a)), (P_n(\beta))) = 1)$$

where f_{ij} is any *quantitative function*.

3.2.1. *Quantitative Functions.* The quantitative functions, $<$, $>$, $=$, $\leq$, $\geq$, min, max, and vac, are functions from 2×2 into 2. $<$, $>$, $=$, $\leq$, and $\geq$, are assigned 1 if the corresponding relation holds, and 0 otherwise, min and max are standard, and the vacuous function, vac, is defined as follows:

$$\forall x, y, \ \text{vac}(x, y) = 1$$

Let us call the set of quantitative functions $\mathbb{Q}$. Let $P_i' : \mathbb{P} \times \mathbb{P} \to \mathbb{Q}$. In other words, to each pair from $\mathbb{P} \times \mathbb{P}$, P_i' assigns a function in $\mathbb{Q}$. We call $P_i'(P_j, P_k)$, f_{jk}.

Definition 3.3. The *property profile* $P_i^*(a * b)$ is defined as,

$$P_i^*(a * b) = \begin{cases} 1 & \text{if } \forall j, k, f_{jk}(a, b) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We call a property profile a *preservational profile* when $i = 1$. The rest of the profiles are collectively called a *non-alethic profile*.

The function *vac* is used to relax the requirements of property profiles. In actual practice, if for some $i, j, f_{ij} = vac$, we omit it from the property profile. vac enables the following classification of connectives relative to the preservational profile.
A connective is said to be *universally restricted* iff

$$\forall i, j, f_{ij} \neq vac$$

A connective is *uniform* iff

$$\forall i, j, f_{ij} = vac \text{ iff } i \neq j$$

A connective is *normal* iff it is uniform and

$$\forall i, j, f_{ii} = f_{jj}$$

A connective is *classical* iff

$$\forall i, j, \text{ if } i \neq 1 \,\&\, j \neq 1, \text{ then } f_{ij} = vac$$

All connectives examined in this study are either classical, normal, or uniform.

f_{11} in the preservational profile determines whether a connective is a conjunction, disjunction, implication, or equivalence. The following table specifies the standard connectives.

$$\text{If } f_{11} = \begin{cases} \text{min} & \wedge \\ \text{max} & \vee \\ \leq & \rightarrow \\ = & \equiv \end{cases}$$

Example 3.4. *Let us consider an example of a non-classical conjunction. Let our element set, E, be a complete collection of binary profile vectors. Let the preservational profile for conjunction be*

$$P_1(a \wedge b) = 1 \textit{ iff } min(P_1(a), P_1(b)) = 1 \textit{ and } P_2(a) = P_2(b).$$

This enables us to construct the root portion of the conjunction matrix.

$\wedge$	*00*	*01*	*10*	*11*
00	*0*	*0*	*0*	*0*
01	*0*	*0*	*0*	*0*
10	*0*	*0*	*1*	*0*
11	*0*	*0*	*0*	*1*

Now suppose we add a non-alethic profile :

$$P_2(a \wedge b) = 1 \textit{ iff } P_2(a) = P_2(b) = 1.$$

Then the completed matrix is

$\wedge$	*00*	*01*	*10*	*11*
00	*00*	*00*	*00*	*00*
01	*00*	*01*	*00*	*01*
10	*00*	*00*	*10*	*00*
11	*00*	*01*	*00*	*11*

The first thing to note is that the conjunction is uniform. The conjunction, however, is nothing like classical. The conjunction could be false (non-designated) although both conjuncts are true (designated). If we attempt to define the disjunction in the standard fashion as $\neg(\neg p \wedge \neg q)$, things become even wilder.

$\vee$	*00*	*01*	*10*	*11*
00	*00*	*10*	*10*	*10*
01	*10*	*01*	*10*	*11*
10	*10*	*10*	*10*	*10*
11	*10*	*11*	*10*	*11*

The disjunction could be true despite the fact that neither of its disjuncts is. If designation signifies something like weak necessity, say, this, of course, comes as no surprise. In the ternary logics of Jennings and Schotch, for example, $\Box p$ and $\Box q$ could be satisfied without $\Box(p \wedge q)$ being satisfied. In addition, in all the usual modal systems, $\Box(p \vee q)$ could be satisfied without either $\Box p$ or $\Box q$ [see [10] and [11]].

3.3. **Unary Connectives.** A unary connective is a function $* : E \to E$. A trivial unary connective is one for which $\forall a \in E, *(a) = a$. Every non-trivial unary connective reverses some of the properties of some of the elements.

Definition 3.5. A unary connective $*$ reverses a property P_i relative to set $E' \subseteq E$ iff for all $a \in E', P_i^*(a) = |P_i(a) - 1|$.

A unary connective $*$ is *uniform*, iff for every property P_i that $*$ reverses, $E' = E$. A unary connective $*$ is a *negation* iff it reverses P_1 and *classical* if it reverses P_1 only and is uniform.

Proposition 3.6. *4-valued Post negation. Post logics can be analyzed as preservationist logics. n-valued logic will require $|\mathbb{P}| = n - 1$. The set E is incomplete in the above sense. For $n = 4$ the set $E = \{000, 001, 010, 100\}$. The following matrix defines Post negation*

α	$\neg\alpha$
000	*001*
001	*010*
010	*100*
100	*000*

Under this analysis, Post negation reverses all of the properties some of the time. Post negation is a negation on our account, since it reverses the root property for some of the cases. For all $n > 2$, however, Post negation is not classical.

4. Paradox-Tolerant Logic

In [5], Jennings and Johnston introduce *paradox-tolerant logic* (hereafter PTL), the first explicitly preservationist implicational logic. PTL was designed to achieve implicational paraconsistency by requiring the implication to preserve additional properties. The implication preserves truth (i.e. designation) and a property that the authors call *fixity*. To represent this addition, the logic uses a complete set of binary profile vectors as its semantic base. An entry (a, b) in the implication matrix receives a designated value iff both properties are preserved, that is $P_1(a) \leq P_1(b)$ and $P_2(a) \leq P_2(b)$. The system uses matrices for conjunction and disjunction, that, as chance would have it, are isomorphic to the matrices of Heyting's intuitionist system when an extra application of Jaśkowski's Γ-function is performed [see [4]]. The negation and disjunction are classical in the above described sense, namely the only property negation reverses is P_1, and $P_1(a \vee b) = \max(P_1(a), P_1(b))$. The negation and disjunction are defined by the following matrices: PTL:

α	$\neg\alpha$
00	10
01	11
10	00
11	01

$\vee$	00	01	10	11
00	00	00	10	11
01	00	01	10	11
10	10	10	10	11
11	11	11	11	11

The other PL connectives are defined in the standard way[1] and are classical as well. In fact, the matrices are characteristic for PL. The main difference is in the implicational connective and the *falsum* constant, which are both independent of the standard PL connectives:

$\rightarrow$	00	01	10	11
00	10	10	10	11
01	01	11	01	11
10	00	00	10	11
11	01	01	01	11

$\bot$
01

As we have already mentioned, PTL's main aim is implicational paraconsistency and paradox-tolerance over implication. This notion of implicational paraconsistency needs some clarification. Ordinarily, a logic L is said to be implicationally paraconsistent if it satisfies

$$\exists\alpha\,\exists\beta \nvdash_L (\alpha \rightarrow (\neg\alpha \rightarrow \beta)) \tag{4.1}$$

If this is the appropriate criterion, then PTL is implicationally paraconsistent. In fact, many additional suspicious PL implicational theorems fail in PTL. To name a few interesting ones,

$$\bot \rightarrow \alpha$$

$$\alpha \rightarrow (\alpha \vee \beta)$$

$$(\alpha \wedge \beta) \rightarrow \alpha$$

all fail. (For a more thorough list see [5]). However, as we have elsewhere noted (see [7]), higher order counterparts of some of these theorems hold in PTL. For instance, all of

$$\bot \rightarrow (\bot \rightarrow \alpha)$$

$$\neg\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$$

$$\alpha \rightarrow (\alpha \rightarrow (\alpha \rightarrow (\alpha \vee \beta)))$$

are theorems of PTL.

This reveals that §4.1 is rather feeble as a criterion of implicational paraconsistency. There is room for higher standards.

Definition 4.1. The implication-negation fragment IN of a logic L is the set of theorems of L that contain no connective other than negation and implication.

[1]The negation is distinct from Heyting's negation and such that disjunction and conjunction are interdefinable.

Then a logic L is said to be properly implicationally paraconsistent iff

(4.2) $\exists\alpha\,\exists\beta$ such that for no $d \in IN_L$, d is of the form $x \rightarrow (y \rightarrow (((\ldots \rightarrow \beta))))$

where x and y range over $\{\alpha, \neg\alpha\}$. Note that §4.2 is weaker than

$$\exists\alpha, \exists\beta, \{\alpha, \neg\alpha\} \nvdash \beta. \tag{4.3}$$

§4.2 could hold of L without §4.3 holding. In the presence of *modus ponens*, the converse is false.

There is a sense, however, in which §4.1 and §4.2 are different ends of a spectrum. To explicate the similarity of the two criteria we need the notion of implicational detonation.

Definition 4.2. Relative to a logic L, a set Σ *implicationally detonates* iff $\exists d \in IN_L$, d is of the form $(x \rightarrow (y \rightarrow (z \rightarrow (((\ldots \rightarrow \beta))))))$ where x, y and z range over elements of Σ, and β is an arbitrary sentence. We call the sentence d an *implicational fuse.*

This is a sense in which the implication of PTL is an improvement over the material implication, although given an inconsistent set both implications are explosive: the theorems driving explosions in PTL need deeper nestings. From the point of view of this investigation, this need for an increase in nesting is a centrally interesting feature of PTL. Furthermore, we can generalize this need for deeper nesting into a paraconsistent measure. According to this measure, the deeper one has to nest to drive an explosion from a given inconsistent set, the more paraconsistent the logic is.

5. Alfred Nobel, and the Need for Finiteness

Once upon a time logicians asked whether logic ought to be descriptive or prescriptive. The former alternative has slipped out of sight not only as the result of the hard philosophical labours of the likes of Gilbert Harman, but also as the response to the empirical findings of the cognitive psychologists and linguists. What remains is the latter alternative. And this alternative gives rise to *inferential deontology.* Inferential deontology is a new branch of philosophical logic which attempts to clarify the prescriptive nature of logic. The inferences are divided into *permissible, impermissible* and *obligatory.* The first theorem of this young discipline is that the goal of classical logic cannot be to set the boundary of obligatory inferences. As a simple demonstration of this fact, we occasionally assign our students the task of drawing all the CL inferences from a finite set of sentences. The reward, an A+ for the course, is received only upon the completion of the task.

The point of the joke is clear. If a logic is to set some class of inferences as obligatory, the class will have to be restricted by finite considerations. Given a finite inferential agent with a limited understanding of the world and a finite amount of time, only so much can be expected.

On the other hand, classical logic may be just what is needed to capture the notion of logical permissibility. In any situation, one is allowed to use any of the classically sound reasoning patterns. It is just that, in the real world, and in real time, one cannot be expected to use all of them. Thus, it might be that where the alternative logics, paraconsistent and other, enter is to capture some refinement of this notion of *logical obligation.* The particular notions of logical obligation will have to capture not only the specifics of the situation, but also the restricted resources of the agents.

Logical obligation may be the additive needed for the somewhat sear garden of paraconsistency. The fact that we all live with inconsistencies among our beliefs is not the result of the unanimous acceptance of dialetheism, but the result of our inability to locate and uncover the inconsistencies. When we do uncover the inconsistencies, however, our first move is not to roll about in them, but to try to resolve them. And when this approach fails, we learn to step around them. It seems fair to say that one of our logical obligations with respect to a set of data is to discover any inconsistencies in it that we are capable of discovering. In this context, some finitely restricted version of *Ex contradictione quodlibet* seems to be our best ally in the discovery of contradictions. What better sign can you have that something is wrong, than that you are permitted to infer anything from your premise set?

This brings us to our next plea to the paraconsistent community, the plea against logical egalitarianism. Not all contradictions are created equal. Although any contradiction permits one certain inferential liberties, some are more deeply promiscuous than others. In fact, from some one is *required* to infer freely, at least so far as to realize that there is a difficulty.

It seems obvious to us that even if some contradictions are to be tolerated, it certainly need not be the case that *all* contradictions are. Even among the ones to be tolerated, if any, some are to be more tolerated than others. A natural way to capture this fact, is to impose a partial ordering on the set of available contradictions. The ordering may, for example, represent complexity of the claims involved. The simpler contradictions explode more readily than the complex and involved ones. The most complex ones, paradoxes, may be tolerated all together. The latter kind are the contradictions that no one is able to resolve in a satisfactory fashion. Some of them may be impossible to resolve, and others may simply be impossible to resolve given the present state of our knowledge.

5.1. **Nitroglycerine and Dynamite.** The story here is not unlike the story of the discovery of dynamite. Nitroglycerine is extremely sensitive to shock. Where nitroglycerine figures as an ingredient in explosives, the kinds of detonating shock have to be distinguished. Adding silica enabled Nobel to distinguish the kinds of shock that ought to initiate explosions, from the ones that ought not. Adding silica progressively softens the explosive potential and requires progressively greater shock to produce explosions. But for any kind of shock, we can add enough silica to prevent the explosion.

If classical logic is nitroglycerine, progressive additions of silica produce various logics in the PTL_n sequence. We call the measure of improvement the Nobel measure.

6. Nobel Measure

The Nobel measure is a measure of how explosive some set of sentences is in a given logic L. The measure concerns the set IN_L, and provides us with the minimally nested implicational theorem required to detonate a given explosive set. If the set is non-explosive then the measure assigns it an arbitrarily high number. The measure depends on several notions, the first one of which is the notion of depth of consequence.

Definition 6.1. *Depth of consequence* is a function $C : \Phi \to Nat$, where Φ is the set of formulae of L. The function is defined recursively as follows:

If α is not an implication sentence, $C(\alpha) = 0$. If α is an implicational sentence, and γ is the consequent of α, then $C(\alpha) = 1 + C(\gamma)$. Thus,

$$C(\alpha \vee \beta) = 0$$
$$C(\alpha \to \beta) = 1$$
$$C(\alpha \to (\beta \to \gamma)) = 2$$
$$C(\alpha \to (\beta \to (\gamma \to \delta))) = 3, \text{ etc.}$$

Definition 6.2. *Fuse-measure.* Let Σ be a set of sentences, and L the logic generated by some system S. Let $\alpha \in IN_L$ be Σ-*detonating* if it is a fuse for Σ. Then the fuse-measure is a function $f^S_\Sigma : IN_L \to Nat, \infty$, such that $f^S_\Sigma(\alpha) = C(\alpha)$, if α is Σ-detonating, ∞ otherwise.

Definition 6.3. *The Nobel measure* is a function $\mathcal{N}_S : \wp(\Phi) \to Nat, \infty$.

$$\mathcal{N}_S(\Sigma) = min f^S_\Sigma(\alpha).$$

Example 6.4. *Material conditional-negation fragment of CL. The shortest implicational fuse for* $\{\bot\}$ *is*

$$\vdash_{CL} \bot \to \beta \tag{6.1}$$

and, hence, $\mathcal{N}_{CL}(\{\bot\}) = 1$.
If the set, however, is $\{\alpha, \neg\alpha\}$, *then the shortest implicational fuse is*

$$\vdash_{CL} \alpha \to (\neg\alpha \to \beta) \tag{6.2}$$

and, hence, $\mathcal{N}_{CL}(\{\alpha, \neg\alpha\}) = 2$.

PTL is an improvement over CL with regard to the Nobel measure.
Neither §6.1 nor §6.2 are theorems of PTL.[2] Hence,

$$\mathcal{N}_{PTL}(\{\bot\}) > 1 \text{ and } \mathcal{N}_{PTL}(\{\alpha, \neg\alpha\}) > 2.$$

The aim of the present study is to construct a sequence of logics $PTL_0 \dots PTL_n \dots$ such that for any inconsistent set Σ,

$$\mathcal{N}_{PTL_0}(\Sigma) \leq \mathcal{N}_{PTL_1}(\Sigma) \leq \dots \mathcal{N}_{PTL_n}(\Sigma) \dots$$

and for $\Sigma = \{\bot\}$ or $\{\alpha, \neg\alpha\}$

$$\mathcal{N}_{PTL_0}(\Sigma) < \mathcal{N}_{PTL_1}(\Sigma) < \dots \mathcal{N}_{PTL_n}(\Sigma) \dots$$

[2] Nor is the obvious modification of §6.2, $\neg\alpha \to (\alpha \to \beta)$.

7. PTL_n Sequence

We want a partial ordering on the set of elements of the system. The ordering represents complexity. Thus, $\alpha < \beta$ represents the fact that β is more complex than α. What we mean by complexity is purposefully left somewhat vague. An easy move is to associate the complexity of a claim with the time it would take a computational device to 'process' it. We do not make this move, although our main idea is not entirely unrelated. On our (admittedly vague) account, a sentence is more complex if it requires deeper understanding of some aspect of the world in order to be processed. What this means is that, in most cases, claims the understanding of which depends upon understanding more of our theories will be more complex. Or, as it is sometimes put, the more complex claims are the ones that depend on more theory.

Each new logic in the sequence distinguishes progressively more elements by their complexity. A natural way to build the sequence of logics capturing such a progression is to use an isomorphism of the lattice ordering of classical truth functions. Each logic in the PTL_n sequence is associated with the lattice defined by $\vdash_{CL}$ over the set of all n-ary connectives.[3] PTL_0 is defined over the set of all CL-constants, PTL_1 over the set of unary connectives, PTL_2 over the set of binary connectives, etc. $\alpha \leq \beta \Leftrightarrow \alpha \vdash_{CL} \beta$. The arity of profile vectors for PTL_n is 2^n.

We define negation, disjunction, implication and countably many constants for the class of lattices as follows:

Definition 7.1. *Disjunction.* If $a \notin D$ and $b \notin D$ then $a \vee b =$ $\max\{\, c \,|\, c \leq a \text{ and } c \leq b\}$. Let $\mathcal{B} : E \rightarrow Nat$ be a function assigning a corresponding natural number to each of the elements. For example, $\mathcal{B}(000) = 0$ and $\mathcal{B}(110) = 6$. If either $a \in D$ or $b \in D$ then $a \vee b = \mathcal{B}^{-1}(\max(\mathcal{B}(a), \mathcal{B}(b)))$.

Definition 7.2. *Negation.* If $a \in D$ then $\neg a = \max\{\, b \notin D \,|\, b \leq a\}$. If $a \notin D$ then $\neg a = \min\{\, b \in D \,|\, a \leq b\}$.

Definition 7.3. *Implication.* The preservational profile of implication is easily defined:

$$P_1(a \rightarrow b) = 1 \text{ iff } a \leq b$$

The issues involved in defining the non-alethic profile add some complication. In [7], we assign the non-alethic profile using a fairly involved algorithm. Here, we sacrifice some of the adherence to the original sequence of logics for the sake of simplicity. The definition will change some of the places in the implicational matrix. Here is the matrix for PTL_1. The items marked with a star have been changed. 0 in the second place has been replaced by 1.

$\rightarrow$	00	01	10	11
00	**10**	**11***	*10*	*11*
01	**01**	**11**	*01*	*11*
10	00	01*	***10***	***11***
11	01	01	***01***	***11***

It is important to note that the new sequence still provides the desired increase in the Nobel measure from one logic to the next.

[3] This approach was suggested to us by D.K. Johnston.

Let E/P_1 be the restriction of the set of elements to non-designative properties, that is, the set of elements without the leftmost property. Let $\mathcal{B}[E/P_1] : E/P_1 \to Nat$ be a function assigning to each element $a \in E/P_1$ the corresponding natural number.

Let a *non-alethic quarter* of a matrix be a segment in which P_1 of the antecedent and the consequent are some fixed values x and y. Then, corresponding to four possible values of x and y—$x = 0$, $y = 0$; $x = 0$, $y = 1$; $x = 1$, $y = 0$; and $x = 1$, $y = 1$—there are four quarters. (See the above table in which the four quarters are emphasized differently.) For every logic in the sequence, the four quarters are identical in E/P_1. Hence, defining the non-alethic profile requires defining only one of the quarters.

Let A and C stand for $\mathcal{B}[E/P_1]$ of the antecedent and the consequent respectively. Let $f(A, C)$ be a function assigning a value to the implication. Then, if $C < n$,

$$A \leq C \Rightarrow f(A, C) = A$$

$$A > C \Rightarrow f(A, C) = C + 1$$

If $C = n$,

$$A < C \Rightarrow f(A, C) = A + 1$$

$$A = C \Rightarrow f(A, C) = n$$

The non-alethic profile is then assigned to implication by $\mathcal{B}^{-1}$

Example 7.4. *Quarters for PTL_1 and PTL_2. As can be checked against the matrix above, the PTL_1 non-alethic quarter is*

A\C	0	1
0	0	1
1	1	1

For PTL_2, it is

A\C	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	1
1	1	1	1	1	1	1	1	2
2	1	2	2	2	2	2	2	3
3	1	2	3	3	3	3	3	4
4	1	2	3	4	4	4	4	5
5	1	2	3	4	5	5	5	6
6	1	2	3	4	5	6	6	7
7	1	2	3	4	5	6	7	7

Definition 7.5. *Constants.* Let $<$ be a linear ordering of $E - D$ by $\mathcal{B}$. Let m be the maximal element on this ordering. Let F be the countable set of false constants $\{\bot, -1, -2, -3 \ldots\}$. Then, if f is the interpretation function, $f(\bot) = m$, $f(-1) = m - 1$, $f(-2) = m - 2 \ldots \forall n \leq -m, f(n) = 0$.

8. Syntax

A formal system of PTL_n is the usual ordered triple

$$S = \langle L,\, A,\, R\rangle$$

where L is a language, A is a set of axioms, and R is a set of rules. As we will see shortly, the only difference between various systems in the PTL_n sequence lies in the set A.
The language L of PTL_n is an ordered triple

$$\langle At,\, k,\, \Phi\rangle$$

At is a denumerable set of propositional variables, k the denumerable set of constants $F \cup \{\neg, \vee, \rightarrow, (,)\}$ where $F = \{\bot, -1, -2, -3, \dots\}$ is a denumerable set of false constants. The set Φ is the usual set of formulae with the addition of the clause

$$F \subseteq \Phi$$

R is a pair. Its only elements are (classical) *modus ponens*, and the rule of substitution. The contents of A for various degrees is determined by the strategy for the completeness proof. In general, $|A_{PTL_n}|$ is $|A_{PTL_0}| + 3n + 1$. That is, the cardinality of some set of axioms for CL, together with an axiom for each meta-valuational property for each of the three connectives. There is an additional axiom for the root property in the case of the implicational connective.

8.1. An Approach to Completeness.

We approach completeness from a literalist point of view. Every semantic property has its syntactic representative in the guise of the above-mentioned set F of false constants. A false constant, or a set of them, enables us to express the presence or absence of a property P_i. If we can perform this kind of translation for every property, then we can use the syntactic translation of property profiles as axioms of the logic.

To begin, we distinguish properties which can be translated into their syntactic correspondents from the ones which cannot be so translated.

Definition 8.1. $P_i \in \mathbb{P}$ is a *recognized property* iff there is $\alpha \in \Phi$, and an interpretation of α such that for some subformula γ in α

$$P_i(\gamma) = 1 \Rightarrow P_1(\alpha) = 1, \text{ and } P_i(\gamma) = 0 \Rightarrow P_1(\alpha) = 0$$

In other words, P_i is recognized iff there is a sentence such that altering P_i for some subformula alone alters its truth value.

Proposition 8.2. *For a preservationist system S, let $E_S[R]$ be the restriction of the element set to recognized properties. Let T_E be the set of S-tautologies over the element set E. That is, the set of sentences that always receive a designated value in S. Then, $T_E = T_{E[R]}$.*

Proof. Obviously, $T_E \subseteq T_{E[R]}$. Suppose there is $\alpha \in T_{E[R]}$ such that $\alpha \notin T_E$. It follows that for some P_i in $E - E[R]$, altering P_i in α alters its truth-value.

An immediate consequence is that for the purposes of axiomatization we can concentrate on translating recognized properties only.

Definition 8.3. A *syntactic representative* ϕ_α for P_i is an element of Φ such that

$$P_1(\phi_\alpha) = 1 \text{ iff } P_i(\alpha) = 1.$$

Proposition 8.4. *Every recognized property has a syntactic representative.*

Proof. Let P_i be recognized, β the sentence that recognizes it, and α the subformula of β which, under some interpretation, changes the truth-value of β when $P_i(\alpha)$ is altered. Note that every element of E can syntactically be represented as either a false constant if not in D, or a negation of a false constant if in D. It is easily seen that, if there is a β that recognizes P_i, then there is a β for which α is a propositional variable. For some such β and for all propositional variables in β except for α, we replace the propositional variables with false constants or their negations according to the dictates of the interpretation. The resulting expression in one variable has the needed property

$$P_1(\phi_\alpha) = 1 \text{ iff } P_i(\alpha) = 1.$$

To prove completeness for any preservationist system for which $|\mathbb{P}|$ is finite, it is sufficient to find a syntactic representative for each recognized property. The translation function, to be described shortly, then enables the translation from the property profiles into axioms. A consequence of the above two propositions is that every finite preservationist system containing a complete set of CL connectives is finitely axiomatizable.

8.2. **Syntactic Representatives for PTL_n.** The preservational profile of implication for any of PTL_n systems is

$$P_1(\alpha \to \beta) = 1 \Leftrightarrow \forall i(1 \leq i \leq n) P_i(\alpha) \leq P_i(\beta)$$

The syntactic representative of $P_1(\alpha) = 1$ is α itself. Let $\bot_i$ be false constant the semantics of which is as follows:

$$P_i(\bot_i) = 1;\ \forall P_j(j \neq i), P_j(\bot_i) = 0$$

For $P_i(1 < i \leq n)$, the syntactic representative of $P_i(\alpha) = 1$ is $\bot_i \to \alpha$.

8.3. **In Search of Axioms.** Thus, $\bot_i \to \alpha$ represents the semantic condition $P_i(\alpha) = 1$. $\neg(\bot_i \to \alpha)$ represents the semantic condition $P_i(\alpha) = 0$. At this point, the reason for making the full force of PL available becomes apparent. Having the force of PL, we can now translate the preservational and the non-alethic profiles directly into axioms of the logic. Those axioms are obviously sound and complete. Using the standard PL connectives,

$$\begin{aligned} p \wedge q &=_{\text{Def.}} \neg(\neg p \vee \neg q) \\ p \supset q &=_{\text{Def.}} \neg p \vee q \\ p \equiv q &=_{\text{Def.}} (p \supset q) \wedge (q \supset p) \end{aligned}$$

we proceed with the translation as follows. Calling the translation function $\mathbb{T}$,

(i)	$\mathbb{T}(P_1(\alpha) = 1) = \alpha$
(ii)	$\forall i(2 \leq i \leq n)\mathbb{T}(P_i(\alpha) = 1) = (\bot_i \to \alpha)$
(iii)	$\forall i, \mathbb{T}(P_i(\alpha) = 0) = \neg(\mathbb{T}(P_i(\alpha) = 1))$
(vi)	$\mathbb{T}(P_i(\alpha) \leq P_j(\beta)) = \mathbb{T}(P_i(\alpha)) \supset \mathbb{T}(P_j(\beta))$
(v)	$\mathbb{T}(P_i(\alpha) = P_j(\beta)) = \mathbb{T}(P_i(\alpha)) \equiv \mathbb{T}(P_j(\beta))$
(vi)	$\mathbb{T}(P_i(\alpha) \geq P_j(\beta)) = \neg(\mathbb{T}(P_i(\alpha))) \supset \neg(\mathbb{T}(P_j(\beta)))$
(vii)	$\mathbb{T}(P_i(\alpha) < P_j(\beta)) = \neg(\neg(\mathbb{T}(P_i(\alpha))) \supset \neg(\mathbb{T}(P_j(\beta))))$
(viii)	$\mathbb{T}(P_i(\alpha) > P_j(\beta)) = \neg((\mathbb{T}(P_i(\alpha))) \supset (\mathbb{T}(P_j(\beta))))$
(ix)	$\mathbb{T}(\min(P_i(\alpha), P_j(\beta)) = 1) = \mathbb{T}(P_i(\alpha)) \wedge \mathbb{T}(P_j(\beta))$
(x)	$\mathbb{T}(\max(P_i(\alpha), P_j(\beta)) = 1) = \mathbb{T}(P_i(\alpha)) \vee \mathbb{T}(P_j(\beta))$
(xi)	$\mathbb{T}((P_i(\alpha)\, vac\, P_j(\beta)) = 1) = \neg(-n)$ (the negation of the least falsum)

Let us look at an example of the application of $\mathbb{T}$. Applying $\mathbb{T}$ to the preservational profile of PTL_1 implication we get,

$$(p \to q) \equiv ((p \supset q) \wedge ((\bot \to p) \supset (\bot \to p))).$$

When, in addition, we translate the non-alethic profile into

$$(\bot \to (p \to q)) \equiv ((\bot \to p) \vee (\bot \to q))$$

we have all the implicational axioms needed for completeness.

The completeness proof itself is a fairly standard Henkin-style construction. Maximal PTL_n-consistent sets are almost entirely standard. The Henkin-model is an $n+1$-tuple consisting of a PTL_n-consistent set, and the n properties which comprise the profile vectors. The fundamental theorem splits into n stages, a stage for each property. We show that

$$\alpha \in \Sigma \text{ iff } P_1^{\Sigma}(\alpha) = 1,$$

and that

$$\forall(2 \leq i \leq n)(\bot_i \to \alpha) \in \Sigma \text{ iff } P_i^{\Sigma}(\alpha) = 1,$$

where Σ is a PTL_n maximal consistent set. The translation function $\mathbb{T}$ ensures that each of the n stages of the proof for each of the connectives proceeds smoothly.

9. Increasing Paraconsistency

Let Th_{PTL_n} be the logic of PTL_n. Since for all n the matrices of PTL_{n-1} are embedable into the corresponding matrices of PTL_n, it is easily shown that $Th_{PTL_n} \subseteq Th_{PTL_{n-1}}$. It follows that for an arbitrary inconsistent set Σ,

$$\mathcal{N}_{PTL_0}(\Sigma) \leq \mathcal{N}_{PTL_1}(\Sigma) \leq \ldots \mathcal{N}_{PTL_n}(\Sigma) \ldots.$$

Showing that for $\Sigma = \{\bot\}$ or $\{\alpha, \neg\alpha\}$

$$\mathcal{N}_{PTL_0}(\Sigma) < \mathcal{N}_{PTL_1}(\Sigma) < \ldots \mathcal{N}_{PTL_n}(\Sigma) \ldots$$

is fairly straightforward. From the function that assigns the non-alethic profile to implication, it can be discerned that $\mathcal{B}(\bot \to \alpha)$ is $\mathcal{B}[E/P_1](\alpha) + 1$ if $\mathcal{B}[E/P_1](\alpha) < \mathcal{B}[E/P_1](\bot)$, and $\mathcal{B}(\neg\bot)$ otherwise. Thus, $\mathcal{N}_{PTL_n}(\bot)$ is $|E_{PTL_n} - D_{PTL_n}|$. It is an easy exercise to show that similar result holds for $\Sigma = \{\alpha, \neg\alpha\}$.

An immediate consequence is that every logic in the PTL_n sequence is finitely implicationally explosive. In other words, there are finite m and l, $m \leq l$ such that $\mathcal{N}_{PTL_n}(\{\bot\}) = m$ and $\mathcal{N}_{PTL_n}(\{\alpha, \neg\alpha\}) = l$.

9.1. **Intersection of PTL_n Logics.** Now, for certain purposes it may be desirable to have a logic in which the Nobel measure for both $\{\bot\}$, and $\{\alpha, \neg\alpha\}$ is ∞. There are two known preservationist ways of achieving this end. The first is to introduce profile vectors of infinite arity. The second, arguably more interesting, approach is to take the intersection of all finite PTL_n cases. That is,

$$PTL_\omega = \cap\{\, PTL_n \mid n \in Nat\,\}$$

An interesting open question in this line of research is the question of finite axiomatization for PTL_ω. The logic is decidable if it is axiomatizable, for every theorem that fails, fails on some finite PTL_n matrix. Since the logic is a sublogic of CL, it would also be of interest to see whether the implication-negation fragment of the logic matches some known paraconsistent sublogic of CL.

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