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Completeness of Indexed ϵ -calculus

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Abstract. Epsilon terms indexed by contexts were used by K. von Heusinger to represent definite and indefinite noun phrases as well as some other constructs of natural language. We provide a language and a complete first order system allowing to formalize basic aspects of this representation. The main axiom says that for any finite collection S_1, \dots, S_k of distinct definable sets and elements a_1, \dots, a_k of these sets there exists a choice function assigning a_i to S_i for all $i \leq k$. We prove soundness and completeness theorems for this system $S_{\epsilon_i}^{fin}$.

1. Introduction

Epsilon calculus was introduced by D. Hilbert in the framework of his program in foundations of mathematics (see [8]). Since then it has proven to be useful in fields as diverse as those of computation and semantic theory of natural language. For instance A. Blass and Yu. Gurevich [3] interpret an epsilon term $\epsilon x \phi(x)$ as a choice construct (choose $x : \phi(x)$) in programs for Abstract State Machines [2]. One of the strategies in [3] allows to make different choices for different occurrences of one and the same ϵ -term. In this sense it is maximally dependent of a context.

The present study derives its motivation from the work in semantic theory, where Urs Egly and Klaus von Heusinger used indexed ϵ -terms $\epsilon_i x \phi$ and their semantic values, indexed choice functions, to represent formally definite and indefinite noun phrases. [See for instance [1], [6], [7]]. This representation exploits two properties of choice functions. First, the element that a choice function assigns to a non-empty set is an arbitrary member of the set. Second, once an element is assigned to a set, it remains fixed. Given these two properties, it is possible to think of indices i as *contexts*. For every predicate α in a given context i , the choice function ϵ_i assigns an object to the predicate: the most salient object in the given context i of which that predicate holds. Thus, for a predicate P in context i , ‘the P ’ is simply the most salient element of P in i , or $\epsilon_i x P x$. Indefinite descriptions,

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on the other hand, are interpreted not in the given context but in a possibly *updated* context. Thus, ‘a P ’ in a context i is interpreted as $\epsilon_j x P x$, where i and j are contexts which possibly differ in what they choose as the most salient element of P . In this paper we devise a formal system we believe is particularly suitable for operating with indexed choice functions.

Updates in our system are all finitely based. Our choice axioms in section 4.2 distinguish a finite number of predicates for a specific update. Nothing is claimed about other predicates: the choice can produce the same elements as in the choice one starts with, or different ones. A change of context will only specify choices from this particular (finite) set of predicates.

In [6] (section 6) indefinite article “a P ” in a context i is modeled using a choice function ϵ_j that is equal to ϵ_i except for the object $\epsilon_j x P x$ assigned to P . In the situation when there are only finitely many sets of immediate interest, given by predicates P_1, \dots, P_n this connection between i and j is easily expressed in our language:

$$i \approx j \equiv \bigwedge_{\beta \in \{P_1, \dots, P_n\}} \epsilon_i x \beta(x) = \epsilon_j x \beta(x) \quad (1)$$

An interesting problem suggested by a referee is to axiomatize the relation $i \approx j$ without any restriction on β by a schema obtained from (1) by changing \equiv to \rightarrow and dropping restriction on β . Most likely, the relation \approx is not axiomatizable, and a Henkin-style partial axiomatization will give the formal system the strength of second order logic.

We introduce the language $L\epsilon$ (Section 2), the deductive system $S\epsilon_i$ and its extension $S\epsilon_i^{fin}$ by finite choice axioms (Section 4.2). Section 3 presents the semantics of choice functions and finite updates. In section 5 we prove the completeness of $S\epsilon_i$ and $S\epsilon_i^{fin}$ with respect to their semantics.

To simplify syntax, we avoid further extension of the language by context ϵ -terms $\epsilon i \alpha$. One of the obvious directions for further work is an investigation of successive update semantics for such an extension as suggested in [5] and by the referee.

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2. The Language of $L\epsilon$

Basic symbols:

Countably many object variables denoted by x, y, z, \dots ; countably many context variables denoted by i, j, \dots ; function symbols (including constants) denoted by f, g, h, \dots for functions from objects to objects; context constants denoted by c, c', \dots ; predicate symbols (including $=$) denoted by P, Q, R, \dots ; propositional connectives \neg and \rightarrow ; indexed epsilon symbols ϵ_a where a is a context variable or context constant; quantifiers over context variables $\exists i$.

Definition 1. (Terms and Formulas)

Context terms: context variables and constants;

Object terms: $x; ft_1, \dots, t_n; \epsilon_a x \alpha$, if x occurs free in α , and a is context term.

Atomic formulae: $Pt_1 \dots t_n$, where t_1, \dots, t_n are object terms; $s_1 = s_2$ where s_1, s_2 are terms of the same kind, i.e. object terms or context terms.

Composite formulae: $\neg \alpha, (\alpha \rightarrow \beta), \exists i \alpha$.

Occurrences of x in α are *bound* in $\epsilon_i x \alpha$.

Familiar notions such as free and bound variable, substitution $s[x/t]$ and $\alpha[i/a]$ for free occurrences of x or i , etc. are defined in a standard way.

We treat formulae up to renaming of bound variables:

$\epsilon_i x \alpha \equiv \epsilon_i y. \alpha[x/y]$, provided y does not occur in α and

$\exists i \alpha \equiv \exists j. \alpha[i/j]$, provided j does not occur in α .

Abbreviations: $\perp, \wedge, \vee, \leftrightarrow, \forall i$ etc. are standard.

Quantifiers over object variables can be introduced in several equivalent ways:

$$\exists x \alpha := \forall i. \alpha[x/\epsilon_i x \alpha]; \quad \exists x \alpha := \exists i. \alpha[x/\epsilon_i x \alpha]; \quad \exists x \alpha := \alpha[x/\epsilon_i x \alpha].$$

For definiteness, we fix a context constant c_0 and define:

$$\exists x \alpha := \alpha[x/\epsilon_{c_0} x \alpha] \quad \forall x \alpha := \alpha[x/\epsilon_{c_0} x \neg \alpha]. \quad (2)$$

3. Semantics

The ϵ -symbol is interpreted as a choice operator: $\epsilon_i x \alpha$ is an x satisfying α if there is such an x , and an arbitrary object, otherwise.

Definition 2. f is a partial choice function for a set \mathcal{S} if $f(S) \in S$ for every non-empty set $S \subseteq \mathcal{S}$ in the domain of f . When f is total, that is defined for all $S \subseteq \mathcal{S}$, it is a choice function for \mathcal{S} .

We say that a set \mathbb{F} of choice functions for \mathcal{S} contains a family \mathbb{F}_0 of partial choice functions if for every $f_0 \in \mathbb{F}_0$ there is an $f \in \mathbb{F}$ such that $f \supseteq f_0$, that is $f(S) = f_0(S)$ when $S \in \text{dom}(f_0)$.

A choice structure is a tuple

$$\mathcal{A} = (|\mathcal{A}|, P^A, \dots, f^A, \dots, \mathbb{F}). \quad (3)$$

where $(|\mathcal{A}|, P^A, \dots, f^A, \dots)$ is a first order structure in a standard sense with object domain $|\mathcal{A}|$, and \mathbb{F} is a non-empty set of choice functions for $|\mathcal{A}|$.

Intuitively, context variables i range over the set \mathbb{F} of choice functions.

Let $OVar$ be the set of object variables $\{x, y, z, \dots\}$, and $CVar$ be the set of context variables $\{i, j, \dots\}$. Let $Var := OVar \cup CVar$.

A variable assignment s assigns elements of $|\mathcal{A}|$ to object variables and choice functions from \mathbb{F} to context variables and context constants.

Note the different treatment of object constants and context constants. Since we need context constants only as witnesses (Henkin constants) in the completeness proofs, they are treated like variables and interpreted by

a variable assignment and not by a structure. The only property of these constants we are interested in is that quantification is not allowed over them. For a variable assignment s we define $\models_{\mathcal{A}} \alpha[s]$ and $\bar{s} : Term \rightarrow |\mathcal{A}|$ simultaneously as follows (skipping the obvious cases for functions and boolean connectives):

$$\models_{\mathcal{A}} Pt[s] \Leftrightarrow \bar{s}(t) \in P^{\mathcal{A}},$$

for $t = t_1 \dots t_n$,

$$\bar{s}(\epsilon_a x \alpha) = (s(a))(\{d \in |\mathcal{A}| : \models_{\mathcal{A}} \alpha[s(x|d)]\}), \quad a \in CVar \cup CConst.$$

where $s(x|d)$ is an assignment equivalent to s on all arguments except x , and $s(x|d)(x) = d$. Similarly for $s(i|f)$.

$$\models_{\mathcal{A}} \exists i \alpha[s] \Leftrightarrow \models_{\mathcal{A}} \alpha[s(i|f)] \text{ for some } f \in \mathbb{F}.$$

A *finite choice* from a family \mathcal{F} of subsets of \mathcal{S} is a partial choice function defined only for finitely many sets belonging to \mathcal{F} .

Definition 3. An *choice structure with finite updates* is a choice structure

$$\mathcal{A} = (|\mathcal{A}|, P^{\mathcal{A}}, \dots, f^{\mathcal{A}}, \dots, \mathbb{F})$$

such that \mathbb{F} contains all finite choices from the family of all subsets of $|\mathcal{A}|$ definable in \mathcal{A} by terms of $L\epsilon$ with parameters from $|\mathcal{A}|$ and \mathbb{F} .

Note that a similar semantics with context variables ranging over all choice functions leads to second order logic. Using a new predicate constant C and object constant 0 one can rewrite a second order comprehension axiom

$$\exists P \forall x (Px \leftrightarrow \phi(x))$$

as

$$\exists i \forall x ((\epsilon_i y C(x, y) = 0) \leftrightarrow \phi(x)).$$

3.1. Substitution of values

Lemma 1. Let s_1 and $s_2 : Var \rightarrow \mathbb{F} \cup |\mathcal{A}|$ agree on all free variables and context constants of a term t and formula ϕ . Then,

$$\bar{s}_1(t) = \bar{s}_2(t) \quad \text{and} \quad \models_{\mathcal{A}} \phi[s_1] \Leftrightarrow \models_{\mathcal{A}} \phi[s_2].$$

Proof. The proof is by a simultaneous induction on terms and formulae. The only case not entirely standard is $t = \epsilon_a x \alpha$, where a is a context term.

$$\bar{s}_1(\epsilon_a x \alpha) = s_1(a)(\{d \in |\mathcal{A}| : \models_{\mathcal{A}} \alpha[s_1(x|d)]\}).$$

By assumption, $s_1(a) = s_2(a)$, and by induction hypothesis,

$$\{d \in |\mathcal{A}| : \models_{\mathcal{A}} \alpha[s_1(x|d)]\} = \{d \in |\mathcal{A}| : \models_{\mathcal{A}} \alpha[s_2(x|d)]\}.$$

□

In particular, for constant terms t ($FV(t) = \emptyset$) and sentences ϕ , 1 holds for all s_1, s_2 that agree on the context constants of t .

Lemma 2.

$$\bar{s}(t[x/u]) = \overline{s(x|\bar{s}(u))}(t),$$

$$\models_{\mathcal{A}} \alpha[x/u][s] \Leftrightarrow \models_{\mathcal{A}} \alpha[s(x|\bar{s}(u))],$$

where x is an object variable and u an object term.

Proof. Simultaneous induction on t, α . \square

4. Deductive systems*4.1. System $S\epsilon_i$*

Axioms:

1. All tautologies, that is, substitution instances of propositional tautologies
2. Equality axioms
 - (a) $t = t$,
 - (b) $t = s \rightarrow (\alpha[v/t] \leftrightarrow \alpha[v/s])$, α atomic,
where t, s are context or object terms of the same kind, $v \in Var$.
3. Critical formulas
 $\alpha[x/t] \rightarrow \alpha[x/\epsilon_a x \alpha]$, provided the object term t is substitutable for x in α , and a is a context term.
4. Extensionality axiom

$$\forall x(\alpha \leftrightarrow \beta) \rightarrow \forall i(\epsilon_i x \alpha = \epsilon_i x \beta).$$

5. Axiom for quantifiers over context variables
 $\alpha[i/a] \rightarrow \exists i \alpha$, where a is a context term.

Inference rules:

Modus Ponens:

$$\alpha, \alpha \rightarrow \beta \vdash \beta.$$

Elimination rule for \exists over indices:

$$\alpha[i/j] \rightarrow \beta \vdash \exists i \alpha \rightarrow \beta$$

with the usual proviso that j does not occur free in the conclusion.

4.2. System $S\epsilon_i^{fin}$

Add the following schema to $S\epsilon_i$:

Existence of a finite choice: for formulas α_k and terms t_k not containing ϵ_i :

$$[\bigwedge_{k \leq n} (\alpha_k[x/t_k] \vee \forall x \neg \alpha_k) \wedge \bigwedge_{k < l \leq n} \exists x \neg (\alpha_k \leftrightarrow \alpha_l)] \rightarrow \exists i (\bigwedge_{k \leq n} (\epsilon_i x \alpha_k = t_k)). \quad (4)$$

Intuitively, the axiom (4) expresses the fact that for any finite collection of distinct sets S_1, \dots, S_n defined by formulae $\alpha_1, \dots, \alpha_n$ and a_1, \dots, a_n respective elements of the sets, there is a choice function Φ such that $\Phi(S_k) = a_k$ for all $k \leq n$. Note that the disjunct $\forall x \neg \alpha_k$ covers the possibility of an empty S_k . For further clarification, we write an instance of the case $n = 2$ in full.

$$((\alpha[x/t] \vee \forall x \neg \alpha) \wedge (\beta[x/s] \vee \forall x \neg \beta)) \wedge \exists x \neg (\alpha \leftrightarrow \beta) \rightarrow \exists i (\epsilon_i x \alpha = t \wedge \epsilon_i x \beta = s)$$

$\vdash \alpha$, $\vdash^{fin} \alpha$ denote derivability in S_{ϵ_i} , $S_{\epsilon_i}^{fin}$ respectively.

$\models \alpha$, $\models^{fin} \alpha$ denote validity in all choice structures and in all choice structures with finite updates respectively.

4.3. Examples

Let's give some examples from the dissertation by Klaus von Heusinger ([5], Section 4.4) devoted to linguistic applications of indexed ε symbol. All these formulas are derivable in $S_{\epsilon_i}^{fin}$ (and as noted by S. Feferman, from a restriction of the schema (4) by $l = 1$), but most of them are not derivable in ordinary ε calculus.

$$\exists x (L(x) \wedge R(x)) \rightarrow \exists i R(\varepsilon_i x L(x)),$$

$$\exists i R(\varepsilon_i x L(x)) \wedge \exists x L(x) \rightarrow \exists x (L(x) \wedge R(x)),$$

$$36a \quad \exists i F \varepsilon_i x F x \iff \exists x F x,$$

$$36b \quad \exists i G \varepsilon_i x F x \rightarrow \exists x G x,$$

$$36c \quad \exists i (G \varepsilon_i x F x \wedge F \varepsilon_i x F x) \iff \exists x (G x \wedge F x),$$

$$37a \quad \exists i F \varepsilon_i x \neg F x \iff \forall x F x, \text{ as well as } F \varepsilon_i x \neg F x \iff \forall x F x,$$

$$37b \quad \forall i G \varepsilon_i x F x \iff \forall x (F x \rightarrow G x) \text{ for } F \neq \emptyset,$$

$$41a \quad \exists i G \varepsilon_i x F x \rightarrow \exists i G \varepsilon_i x [F x \wedge G x], \text{ d}$$

$$51 \quad \forall i [\exists j [G \varepsilon_i x F x \rightarrow G \varepsilon_i x [F x \wedge G x]],$$

$$56 \quad \forall i \exists j [G \varepsilon_i x F x \rightarrow \varepsilon_j x F x = \varepsilon_j x [F x \wedge G x] = \varepsilon_i x F x],$$

$$57 \quad \exists x (F x \wedge G x) \rightarrow$$

$$\forall i \exists j [G \varepsilon_i x [F x \wedge G x] \rightarrow \varepsilon_j x F x = \varepsilon_j x [F x \wedge G x] = \varepsilon_i x [F x \wedge G x]],$$

$$59 \quad \forall i [\exists x (F x \wedge G x) \rightarrow F \varepsilon_i x [F x \wedge G x]].$$

5. Soundness and completeness theorems

S_{ϵ_i} , $S_{\epsilon_i}^{fin}$ are sound and complete in their semantics.

Theorem 1. $\vdash \alpha$ implies $\models \alpha$; $\vdash^{fin} \alpha$ implies $\models^{fin} \alpha$.

Proof. Induction on deduction. Finite choice axioms are realized by extensions of finite choice functions

Theorem 2. (*Completeness theorem*).

$\models \alpha$ implies $\vdash \alpha$; $\models^{fin} \alpha$ implies $\vdash^{fin} \alpha$.

Proof. The proof uses a standard Henkin-construction. As in the case of FOL, we define $\Gamma \vdash \alpha$, $\Gamma \models \alpha$ for arbitrary sets of formulae in a standard way and prove that every consistent set Γ (that is Γ such that $\Gamma \not\vdash \perp$) has a model \mathcal{A} , $\models_{\mathcal{A}} \Gamma$ (that is, $\models_{\mathcal{A}} \gamma$ for all $\gamma \in \Gamma$).

Where possible, we follow closely the model construction steps of the first-order completeness theorem [see, for instance, [4] Chapter 2, Section 2.5]. We deal with quantification over context variables like in the usual first order case.

Step 1. [cf. steps 1-3 in [4] Chapter 2, Section 2.5] We begin with a consistent set Γ . Renaming context constants in Γ if necessary, we ensure that there is a sufficiently big set C of context constants not occurring in Γ . We extend Γ to a maximal consistent set Δ containing witnesses from a set C of new constants: for any context variable i , and any formula α , there is a constant $c \in C$ such that

$$(\exists i \alpha \rightarrow \alpha[i/c]) \in \Delta.$$

For instance, if α is $P\epsilon_i x Qx$ then

$$(\exists i P\epsilon_i x Qx \rightarrow P\epsilon_c x Qx) \in \Delta.$$

We ensure that Δ contains witness formulae, by extending the language $L\epsilon$ to a language $L\epsilon^+ = L\epsilon \cup C$. The proof of consistency of Δ in $L\epsilon^+$ replicates the usual proof for first-order logic.

Proposition 1. Δ is deductively closed, that is, if $\Delta \vdash \alpha$ then $\alpha \in \Delta$.

Proof. Standard [cf. [4] pp. 129].

We need to show that Δ contains witnesses of the more usual sort:

Proposition 2. For all $\alpha, x \in OVar$, $a \in CVar \cup C$, either $\alpha[x/\epsilon_a x] \in \Delta$ or $\neg\alpha[x/t] \in \Delta$ for all t .

Proof. Since formulae $\alpha[x/t] \rightarrow \alpha[x/\epsilon_a x]$ are axioms, if $\neg\alpha[x/\epsilon_a x] \in \Delta$ for some a , then $\neg\alpha[x/t] \in \Delta$ by 1. \square

Step 2. We turn Δ into a structure \mathcal{A} in a standard first-order way. We form a quotient structure which respects equalities in Δ .

Let \mathcal{A}_0 be the set of all closed object terms of the language $L\epsilon^+$. We define the domain $|\mathcal{A}|$ by $|\mathcal{A}| := \{[t] : t \in \mathcal{A}_0\}$ where $[t] := \{u : t = u \in \Delta\}$. \mathcal{A} is then defined as follows.

For each n -ary predicate symbol P , define

$$([t_1], \dots, [t_n]) \in P^{\mathcal{A}} \Leftrightarrow Pt_1 \dots t_n \in \Delta.$$

For each n -ary function symbol f , define

$$f^{\mathcal{A}}([t_1], \dots, [t_n]) = [ft_1 \dots t_n].$$

Let \mathbb{I} be the set of context terms of the extended language, and for $a \in \mathbb{I}$, put $[a] := \{a' : a = a' \in \Delta\}$. Let f be some fixed choice function for $|\mathcal{A}|$. We define for $a \in \mathbb{I}, S \subseteq |\mathcal{A}|$:

$$\Phi_{[a]}^A(S) = \begin{cases} [\epsilon_a x \alpha] & \text{if } S = \{[t] : \alpha[x/t] \in \Delta\}, \\ f(S) & \text{otherwise.} \end{cases}$$

Let $\mathbb{F} := \{\Phi_{[a]} : a \in \mathbb{I}\}$.

Next we need to prove that the structure is well-defined. For the main part the proof is standard. In addition to showing that predicates and functions are well defined, we need (for the fin -case) to show that \mathbb{F} contains all finite updates.

Proposition 3. 1. $\Phi_{[a]}$ depends only on $[a]$

2. $\Phi_{[a]}$ is a choice function for every $a \in \mathbb{I}$

3. $\Phi_{[a]}(S)$ does not depend on the choice of the defining formula for S

Proof. (1). Let $a, a' \in \mathbb{I}$ and $(a = a') \in \Delta$. Then $(\epsilon_a x \alpha = \epsilon_{a'} x \alpha) \in \Delta$ by the equality axioms and so $[\epsilon_a x \alpha] = [\epsilon_{a'} x \alpha]$, hence $\Phi_{[a]}(S) = \Phi_{[a']}(S)$ for every $S \subseteq |\mathcal{A}|$.

(2). Let $S \subseteq |\mathcal{A}|$ be non-empty. If $S = \{[t] : \alpha[x/t] \in \Delta\}$ for some α , then $\alpha[x/t_0] \in \Delta$ for $t_0 = \epsilon_a x \alpha$ by Proposition 2, hence $\Phi_{[a]}(S) = [t_0] \in S$. If S is not definable, then $\Phi_{[a]}(S) = f(S) \in S$, since f is a choice function.

(3). Let $S = \{[t] : \alpha[x/t] \in \Delta\} = \{[t] : \beta[x/t] \in \Delta\}$. Then, $\forall x(\alpha \leftrightarrow \beta) \in \Delta$. Suppose not. Then, by maximality of Δ , $\exists x \neg(\alpha \leftrightarrow \beta) \in \Delta$. This by definition is $\neg(\alpha[x/\epsilon_{c_0} x \neg(\alpha \leftrightarrow \beta)] \leftrightarrow \beta[x/\epsilon_{c_0} x \neg(\alpha \leftrightarrow \beta)]) \in \Delta$, see (2). By deductive closure, $\alpha[x/\epsilon_{c_0} x \neg(\alpha \leftrightarrow \beta)] \in \Delta$ iff $\beta[x/\epsilon_{c_0} x \neg(\alpha \leftrightarrow \beta)] \notin \Delta$, which contradicts the assumption. By the extensionality axiom, $\epsilon_a x \alpha = \epsilon_a x \beta \in \Delta$, and hence, $[\epsilon_a x \alpha] = [\epsilon_a x \beta]$. \square

Proposition 4. In the case of $S\epsilon_i^{fin}$, the set \mathbb{F} is closed under finite choices over the definable subsets of $|\mathcal{A}|$.

Proof. By the deductive closure of Δ all instances of the existence of finite choice are in Δ . For distinct non-empty sets S_1, \dots, S_l defined by formulas $\alpha_1, \dots, \alpha_l$ consider the witness c in the formula $\exists i(\bigwedge(\epsilon_i x \alpha_k = t_k)) \rightarrow \bigwedge(\epsilon_c x \alpha_k = t_k)$. Then $\Phi_{[c]}$ is the required choice function. \square

Definition 4. Grade $GR(e)$ of an expression e , (that is, the number of occurrences of connectives and ϵ in e).

$$GR(x) = GR(c) = 0;$$

$$GR(Pt_1 \dots t_n) = GR(ft_1 \dots t_n) = \max(GR(t_1), \dots, GR(t_n));$$

$$GR(\neg\alpha) = GR(\alpha) + 1; GR(\alpha \rightarrow \beta) = \max(GR(\alpha), GR(\beta)) + 1;$$

$$GR(\epsilon_a x \alpha) = GR(\exists i \alpha) = GR(\alpha) + 1.$$

Lemma 3. Let s be a variable assignment for $|\mathcal{A}|$ such that $s(c) = \Phi_{[c]}$ for every context constant c . Let $[\bar{x}/\bar{t}]$ be a substitution with $s(x) = [x[\bar{x}/\bar{t}]]$ and $s(i) = \Phi_{[i[\bar{x}/\bar{t}]]}$ for all free object variables x and context variables i of a term u or a formula α . Then

1. $\bar{s}(u) = [u[\bar{x}/\bar{t}]]$ for object terms u ,
2. $\models_{\mathcal{A}} \alpha[s]$ iff $\alpha[\bar{x}/\bar{t}] \in \Delta$,
3. $\bar{s}(u) = \Phi_{[u[\bar{x}/\bar{t}]]}$ for context terms u .

Proof. By a simultaneous induction on the grade. Let e be u or α .

Basis $GR(e) = 0$. Then e is either a free variable, constant, an atomic formula $Pt_1 \dots t_n$ or $ft_1 \dots t_n$ with t_i being ϵ -free for every i .

Immediate.

Inductive step. When e is a formula of the form $\neg\alpha$ or of the form $\alpha \rightarrow \beta$, the treatment is standard.

Let $u = \epsilon_a v \alpha$. We have

$$\begin{aligned}
\bar{s}(\epsilon_a v \alpha) &= \bar{s}(a)(\{[t] \in |\mathcal{A}| : \models_{\mathcal{A}} \alpha[s(v|[t])]\}) \\
&= \Phi_{[a[\bar{x}/\bar{t}]]}(\{[t] \in |\mathcal{A}| : \alpha[\bar{x}/\bar{t}, v/t] \in \Delta\} \text{ by (3),(2)}) \\
&= \Phi_{[a[\bar{x}/\bar{t}]]}(\{[t] \in |\mathcal{A}| : \alpha[\bar{x}/\bar{t}][v/t] \in \Delta\}) \\
&= [\epsilon_a[\bar{x}/\bar{t}] v . \alpha[\bar{x}/\bar{t}]] \\
&= [(\epsilon_a v \alpha)[\bar{x}/\bar{t}]]
\end{aligned}$$

$e = \exists i \alpha$. Assume $\exists i \alpha \in \Delta$. Then, by deductive closure and the existence of witness formulae, $\alpha[i/c] \in \Delta$ for some c . But then, $\models_{\mathcal{A}} \alpha[(i|\Phi_{[c]})]$, hence $\models_{\mathcal{A}} \exists i \alpha$.

Assume $\exists i \alpha \notin \Delta$. Then, for every a , $\alpha[i/a] \notin \Delta$. By Induction Hypothesis, for every a , $\not\models_{\mathcal{A}} \alpha[(i|\Phi_{[a]})]$. Hence, $\not\models_{\mathcal{A}} \exists i \alpha$. \square

We see that \mathcal{A} is a model of Γ by 3(2).

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