Modal Logic for Qualitative Dynamics

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> This qualitative study, once completed, will be of the greatest utility to the numerical calculation of the function. Furthermore, this qualitative study will be in itself, of primary interest. Many important questions in Analysis and Mechanics in fact reduce to just this.
>
> *Henri Poincaré*

1 Introduction

The goal of the present study is to introduce a general formalism in which different dynamic modal logics can be compared and categorized. Our analysis will conform to the relatively standard analysis of dynamical systems that dates back at least as far as Henri Poincaré's work on the three-body problem at the end of the 19th century. The modern incarnation of this study, the theory of complex systems, that these days includes chaotic and nonlinear dynamical systems as well as their better behaved cousins, linear systems is among the hottest scientific pursuits. The top prizes in this pursuit including the understanding of multicellular organisms, ecosystems, social species such as ants, bees, and primates (humans), social and economic complexes. As the name indicates, the theory of *complex* systems, the study of such objects is a difficult endeavor. We as a community of scientists, logicians, and philosophers need all the help that we can get, and as many diverse points of view as we can muster. Our systematic approach to dynamic modal logics has as one of its aim bringing logical approaches one small step closer to the research community that studies complex dynamics. Once we can see various logical systems as instantiations of the

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same dynamic set of phenomena, we can not only compare the logics among themselves, but we can also participate in an active exchange of results between now numerous fields that study various aspects of complex dynamics: from mathematics, physics, chemistry, and biology, all the way up to computer science, economics, sociology and anthropology. We will use Iterative Function Systems (IFS), a concept familiar from various approaches to dynamics, as the underlying system in which, we will claim, a number of interesting modal logics can be fruitfully interpreted.[1] Furthermore, modal systems interpreted in IFS, we will argue, mostly instantiate what is known in the dynamical system community as *local* perspective. We will for comparison present a *global* modal system and argue that a class of such global systems is important to the modal study of dynamics, not only for its computational advantages, but also for the readily available and fully transparent view of dynamics.

We will use somewhat freely concepts borrowed from the theory of dynamical systems, mostly from its mathematical aspects. Strogatz [17] Thompson and Stewart [16] provide an excellent introduction to the field of complex dynamics. We will mostly use basic concepts and explain the ones we use, but if further reference is needed, the two books should provide sufficient background. On the modal logic side, we assume that the reader is familiar with standard modal logic, and at least to some extent, Dynamic Epistemic Logic. Nothing that could have not been gained through a careful study of [3] will be needed.

1.1 Modal View of Dynamics

Phase spaces, formal models of change in dynamical systems, cry out for an interpretation in the language of modal logic. If there were no other reasons, and there are plenty, one would set out to interpret phase spaces modally just because of the commonplace claim in the philosophical literature that such dynamic models are somehow essentially superior to the more standard possible world models.[2] In our view, the philosopher's conviction notwithstanding, the *phase space* is really only slightly peculiar when viewed as a frame of modal logic. It is peculiar only in that it combines Kripke and topological modal semantics in a single frame.[3] In the standard phase spaces, we have a topological space

[1]The conception of an Iterative Function System is somewhat more liberal than that of the computer scientist Michael Barnsley who popularized the usage in [2]. We decided to stick with this name for we found it the most evocative of the role that such systems play in modal logic.

[2]For instance, Manuel De Landa in his 2002 book *Intensive Science and Virtual Philosophy* makes such a claim at the end of chapter 1. He sees dynamical systems, phase spaces, and vector spaces as deeply metaphysically distinct from and preferable to the possible world approach to formal metaphysics.

[3]This combination of the two modal semantics has been introduced quite independently from any dynamic considerations. For instance in various combinations of modal logics such a products or fusion models, it has become commonplace to have both Alexandroff and metric topologies alongside each other. The former is just a transitive, reflexive Kripke frame. Results about their interaction on a single frame

and a single change tracking function f, both of which are readily amenable to various modal interpretations. The topology will interpret a variety of topological modalities, while the function f, as a special case of a binary relation will interpret a variety of Kripke style relational modalities. Our main goal here is to explore modal languages that have been used for interpreting the function f and the various topological properties of the underlying metric space modally. Our main contribution is the introduction of a class of novel qualitative dynamical modalities that in our opinion have an essential role to play in the modal approach to the theory of the dynamical systems. As we hope the presentation will make clear, our language readily generalizes to a number of more specific kinds of function systems that extend or generalize IFS, the general setting for modal thinking about dynamics we introduce next.

2 Iterated Function Systems and Some General Notes on Dynamical systems

We need a general mathematical description of a *Complex Dynamical System* that will enable us both to systematize the taxonomy of the existing Dynamical Modal Logics and Motivate introduction of new classes of dynamic logics. The mathematical structure that we propose below, IFS, in our view strikes a healthy balance between including as large and as feasible a class of logical systems that claim to be dynamic, and respect for the historical usage of the term 'dynamical' in mathematical physics where the term originated.

Let X be some metric topological space, and let $\mathcal{F} = \{f_1, ..., f_n\}$, $f_i : X \to X$ for $i \in \{1, ...n\}$ be a set of functions on X.

Definition 2.1 (IFS) *We call* $\mathfrak{X} = (X, \mathcal{F})$ *an* iterated function system*, or* IFS *for short.*

In the simplest case, $\mathcal{F}$ is a singleton function. In such a case, we write $\mathfrak{X} = (X, f)$.

Example 2.2 *For a simple but interesting example, take for instance the closed interval* $[0, 1] \subseteq \mathbb{R}$ *as the underlying metric space, and the* Tent Map *as the time function* f*. Tent map is defined as follows:*

$f(x) = rx$ *if* $x < \frac{1}{2}$ *and* $f(x) = r(1 - x)$ *otherwise.*

For different values of $r \in (0, 2]$ the behavior of f varies wildly. For instance, at $r = 0.75$, the point 0 acts as an attractor. If we think of time in the IFS as a repeated application of the function f, then over time for any $x \in [0, 1]$, the orbit $x, f(x), f(f(x)), f(f(f(x))), ...$ will converge towards 0. Take for instance $x = 0.6$. The sequence is then $0.6, 0.3, 0.225, 0.16875, 0.1265625, 0.095$ and so on. The sequence clearly approaches 0. The choice of 0.6 was arbitrary. We could

are also fairly common these days.

have started anywhere in the interval as the point 0 acts as an attractor in the system.[4] Note here that we can describe the global behavior of our IFS by simply saying that all orbits, wherever they start, tend towards 0, a pretty concise summary of the dynamics.

If we however set $r = 2$, the behavior becomes much more complex. Now behavior of the IFS becomes *chaotic* in the technical sense. Roughly, taking an average orbit, it will over time come close to *any* point in the interval. The orbit is thus bouncing back and forth around the interval. *Average* is an important term here, as many orbits, in fact countably many of them, will behave in an orderly fashion. For instance if we start with $x = \frac{2}{3}$, the orbit infinitely repeats $\frac{2}{3}$ as $2(1 - \frac{2}{3}) = \frac{2}{3}$.

A good intuitive feel for why IFS provides a good model of dynamical systems is provided by Conway's game of life. Although the underlying space Conway uses is different, the idea behind the game is the same. One has a space and a set of change functions. A nice animation of the dynamics of hexagonal variant of Conway's game that brings the dynamics to life, can be seen at:

`http://en.wikipedia.org/wiki/File:Oscillator.gif`.

2.1 Time and Space as Dynamical Control Variables

2.1.1 Kinds of Time

Let $\mathfrak{X} = (X, f)$ be an IFS. We can categorize classes of IFS systems according to the properties of f. Thus for example an IFS can be continuous, open, homeomorphic, interior, affine, linear, etc. as reflected by the properties of f. For instance, one would use linear transformations when one is interested in preserving lines but not, say, angles. If one is not interested in preserving lines, but only general shape properties of an object such as connectedness, one would use homeomorphisms.

A further important distinction is between *deterministic* and *stochastic* IFS. The terminology is borrowed from the theory of dynamical systems and differential equations.[5] The two concepts are meant to capture the difference between the kind of change where the next state of the system is rigidly determined, versus the kind of change that can result in one of finitely many states, each with some likelihood. For instance, every time you press the same key on your computer's word processing program, the same symbol appears on the screen. This would be a clear case of deterministic dynamics. In contrast, every time you toss a dice, one of six different options happens, each one in this case with the

[4] See below for a detailed discussion of attractors.

[5] We mention the distinction between differential and difference equations throughout the paper. The difference is, in short, that differential equation model time as a continuum–time is modeled on a real line $\mathbb{R}$, while the difference equation model discrete time–time is modeled as the natural numbers $\mathbb{N}$.

probability $\frac{1}{6}$. This is a case of a stochastic process. Bellow are formal renderings of this difference.

Definition 2.3 *2.1.2 An IFS $\mathfrak{X} = (X, f)$ is said to be* deterministic *if for all $x, y \in X$, if $x \neq y$ then $f(x) \neq f(y)$.*

In formal speak, f is nonconvergent. As we have seen in the example of the keyboard, a deterministic IFS models systems that do not change the underlying 'atoms' of the space. Spatial relationships between *atoms* change overtime, but the parts remain the same. If you for instance drive your car around town, the matter that the car is made of remains (largely) the same, but what changes are relationships among different parts. Wheels turn, engine parts move, etc. Or for a different example, if you are molding clay, the piece of clay with its molecules remains the same, but the positions of different molecules change over time. Thus the idea of a deterministic IFS is based on a physical intuition according to which the 'number' of basic parts, atoms, molecules, remains the same while their topological and geometric arrangements become more or less intricate over time. Furthermore, for any arrangement X at time k, the function f determines the exact arrangement at any subsequent time $k + l$. There is no indeterminism in the system.

The stochastic model retains this basic intuition that the number of spatial parts remains the same, but allows for nondeterministic change. We cannot tell beforehand which of the several available states the current state will be transferred into. Thus, dice toss cannot be modeled deterministically as we can't tell which of the faces the dice will land on. One is to imagine the change function f specifying that $f(x)$ is 1 with the probability $\frac{1}{6}$, 2 with the probability $\frac{1}{6}$, etc.

Formally, let P be a standard probability measure. For a sentence A, we write $P(A) = r$, where $0 \leq r \leq 1$.

Definition 2.4 *An IFS $\mathfrak{X} = (X, f)$ is a* stochastic *IFS iff for all $x \in X, \exists y_1, ..., y_n$, $P(f(x) = y_1) = q_1, ..., P(f(x) = y_n) = q_n$ and $q_1 + ... + q_n \leq 1$.*

Rather than assigning a fixed successor state y to any state x, the stochastic IFS assigns a finite set of 'possible' successors $y_1, ...y_n$ to each state x. Each successor is reached with nonzero probability and probability that one of the successors is reached is 1. The two clauses simply ensure that the transitions are well behaved probabilistically. In a stochastic IFS, f is best thought of as a relation rather than function and the vertices $(x, f(x))$ are weighted. Each x has at least one relatum y, no more than finitely many relata, and the sum of the weights of the vertices is 1. Thus a stochastic IFS is really modal logical frame (X, R) with the serial, finitely branching, weighted relation R. If successors are equiprobable, we do just have the standard modal logical framework.

In this study, we will mostly be dealing with deterministic IFS's with a single iterative function f and an underlying metric space. An occasional remark may be made about other kinds of spaces, but their systematic study is largely left untouched here. We think that the stochastic IFS's are definitely worth studying for their own modal logical merits. For instance, even the question of what would count as a suitable modal language for expressing interesting properties of stochastic IFS's is nontrivial. Even the idea of weighted Kripke frame is worth some serious attention. Part 1 of this volume contains a number of important contribution to the study of time in dynamic logic, but also in our thinking about time in dynamical systems in general. Part 3 contains some important contributions on the stochastic notion of time and the its role in dynamic thinking.

2.1.2 Some Spatial Variations

For some time now, the wider dynamical system community has recognized that the structure of space matters.[6] The examples where underlying space influences the diversity of dynamical behaviors in the model abound. For instance, in the dynamical models of evolution, paying attention to the intricacies of spatial patterns has proven useful. A long standing puzzle of how a species evolves reproductive self control has recently been solved using spatial tools. Essentially the researchers realized that a sufficiently high level of spatial segregation among the members of the same species enabled a subgroup that was only moderately eager in their reproductive practice to survive while the main overly eager group went extinct as the result of the environmental over-exploitation. In researchers' own words, using spatial techniques in the model enabled them to show: "how spatially distributed populations avoid overexploiting resources due to the local extinction of over-exploitative variants..."[7]

Another striking example from evolutionary biology goes in the opposite direction: rather then disconnecting subgroups, the setting insists on connectedness. Random acts of kindness on their own have been shown insufficient for the evolution of altruism. If, however, an adequate number of such acts are peppered in the space with an appropriate structure, that is, a space that is well-connected, altruism becomes an evolutionarily stable strategy. The structure of space helps us be niceer to each other!

Returning to our exploration of properties of IFS, there are several important kinds of spaces that do not fit our description of the IFS built on a metric topology, but which can be easily accommodated if necessary. First, the requirement that the topology be metric is rather strong and it is often relaxed in the actual study of dynamical systems.

[6]The importance of space in dynamics was first brought to my attention by J. van Benthem and A. Baltag.

[7]See work of Bar Yam at the *New England Institute for Complex System* for the research on the importance of spatial patterns in understanding of evolution. In particular their recent *Nature* paper [5] describes the research we mention here. The group also studies the importance of spatial patterns in a variety of other kinds of dynamical systems. Examples range from negotiation of tasks, to large scale design projects, such as design of cars or airplanes.

There is a large variety of non metric spaces, such as network spaces, lattices, and various Kripke spaces, that are of great significance in the study of complex systems, both for their simplicity and their structural symmetry. Second, there is no reason to limit oneself to just one topology per space. Some of the most successful dynamical logics consist of more than one topology. A familiar example is the Dynamic Epistemic Logic which represents epistemic possibilities for agents as (S4) or (S5) structures which are essentially transitive reflexive Kripke structures and hence fall in the class of Alexandroff spaces. We discuss DEL as an IFS below in detail. Third, there is the singleton topology. This choice of topology is essentially a signal that spatial properties can be ignored. They add no interesting dynamics to the system. Finally, as a negative observation, it is important to note that commonly when one deals with high density grids or lattices in computational representations of dynamics, one often does so simply as a reasonable approximation of some standard metric space and not out of some strong theoretical commitment.

To sum up, occasionally it is well worth relaxing the IFS requirement on spatial properties of the dynamical systems, especially to include graph-like structures and multiple interacting topologies. Graph-like structures often provide one with interesting insight into dynamical systems even when the structure of the graph falls short of meeting the standard of a metric topology or even general topology. Furthermore, as we are sometimes interested in multiple layers of conceptual structure, representing several layers of properties by allowing for several topologies in one system often proves fruitful.

2.2 Time, Change, and Dynamics

Intuitively, we think of iterative function systems as temporally evolving systems. Formally, they are essentially systems of *difference equations*. For any given state x in our underlying topological space X and any starting time t, the deterministic IFS gives us a unique *future* state of the system. Thus we can think of f as determining the unique step by step evolution of the dynamical system $\mathfrak{X}$. For any topological, geometric, or even randomly assembled object $O \subseteq X$, f gives us the unique trajectory of O in time. In particular it tells us what O 'looks like' after n steps for any n. The possible changes of O thus depend on the kind of function that f is. If f is for instance a homeomorphism, and O is say a doughnut–to use the well worn topological example– then $f^n(O)$ is some topological equivalent of the original doughnut O. It could, say, be a cup with a single holed handle, to continue the overused example, but it could not be a bottle (without a handle). Different maps would enforce stronger or weaker relation between O and $f^n(O)$. If f is an arbitrary map, then we could not predict any particular property of $f^n(O)$ including persistence in time, and if f is a rigid transformation, then we could say a fair bit about O's geometrical features after n stages of f assuming that O was reasonably geometrically coherent to begin with. We are mostly interested in well behaved maps that are at the very minimum continuous, but one could sensibly deal with a much wider range of functions. The particular application determines the strength of the function f and the properties preserved by f over time. It is a quite curious observation that in thinking about the world dynamically, the features of time are in some sense determined by our particular goals in exploring the given system. If we are modeling a spatial system, but we are only interested in its topological properties, then our time will be represented by some "topological" function. Time will be a continuous function or a homeomorphism. The more spatial details we are interested in exploring, the stronger the function f representing time. Putting things in terms of change rather than time, the stronger the function f, the less change it allows over some fixed period of time. Thus a function f that is a rigid transformation will never allow a ball to be transformed into a box without loss of identity. Thus, the change allowed in a class of objects will determine the way the properties of time apply to that class of objects. The claim here need not be taken overly metaphysically. Simply, the IFS model in which f is a strong geometrical function will be a rather lousy model for a temporal evolution of Play-Doh on a desk of a kindergartener. Such an IFS will hopefully be a pretty good model for changes that your car undergoes during a normal week of operation. Hopefully no nonrigid transformations, squishing or mangling are taking place. There is a lot more we can say about the time/change function f of an IFS. In some sense, continuity is an external property of time function f. Continuity tells you how time interacts with space, but time has inherent features too. For instance, time can be discrete, dense, continuous, or even finite.

2.3 CFS: Time as Continuum

IFS is the preferred view of dynamical systems in computer science. Natural representation of time in a computer is discrete; one could of course simulate continuous quantities of time via some discrete approximation, but why not just be honest and realize that the dynamics in a digital computer is always discrete. The main contrast class comes from physical and mathematical sciences where dynamics and time are most often viewed as a continuum. We can call this approach CFS approach, standing for *continuous-time function systems*. The two approaches are very closely related and to a large degree complementary. In our view, it bears fruit to explore and understand them side by side. CFS too, may be defined over a metric space X as follows. Let $\mathcal{F} = \{f_1, ..., f_n\}$, $f_i : \mathbb{R} \rightarrow X$ for $i \in \{1, ...n\}$ be a set of differentiable continuous functions onto X, then:

Definition 2.5 (CFS) *We call* $\mathfrak{X} = (X, \mathcal{F})$ *a* continuous-time function system–CFS *for short.*

Thus, CFS is essentially a system of differential equations. The main difference between an IFS and a CFS is in their respective notions of time. In an IFS or a difference system, time is discrete. It makes sense to talk about the initial time, and then a sequence of discrete times that follow the initial moment. The functions f_i are said to order the set X

temporally, and for $r < q$, $[f_i(r), f_i(q)]$ is a closed temporal interval. If we have more than one function in $\mathcal{F}$, then the notion of time is nondeterministic.

Example 2.6 *In IFS it makes sense to say:*

i) In the fifth stage of the evolution of the dynamics of X, the system was stable. In the sixth stage an event B happened and it destabilized the system X in the stage immediately following that one.

In CFS, the time is continuous and the notion of 'next moment' does not make sense. Instead of (i), we could say:

ii) the dynamics of X was stable for a while, but it then destabilized shortly following an event e at $f(r)$.

It is worth noting that there are standard ways of 'translating' the difference equation systems to differential systems, but as we know from say temporal logics, the continuous nature of time introduces further interesting formal complications. Thus the move from discrete or even dense time to the continuous time is certainly not trivial. What is often gained, however, is a certain amount of smoothness in the formalism itself. More about the difference between discrete and continuous dynamic logics below.

Example 2.7 *For an additional example of the contrast between IFS and CFS consider the difference between the changes in the amount of money in your bank account against the changes in your body weight (expressed in kilograms). Your body weight is a paradigmatic physical quantity that is changing continuously in time. If today you weigh 99 kg and yesterday you weighed 97 kg, a wild change indeed, you have transitioned smoothly from 97 to 99 kg. Another way to put this is that if at time t_1 your weight was 97 kg and at t_2 it was 99 kg, then for any weight w between 97 kg and 99 kg there was some time t', $t_1 \leq t' \leq t_2$, and your weight was w at t'. Put yet another way, there was no sudden jumps in weight over time, but you rather transitioned smoothly from $97kg$ toe $99kg$.*[8]

The amount of money in your bank account, in contrast, consists of a discrete series of 'jumps'. As withdrawals alternate with deposits, the total amount in your account jumps from an old total to a new total. The change is sudden and momentary, and no smooth transition takes place. The latter process can be modeled quite appropriately as a sequence of discrete moments and the totals at any such moment.

The former process seems to be more amenable to modeling smoothly in continuous time. One could certainly model one's weight changes in discrete time and measure, say, daily. In fact, practicality may require one to do so. One would be hard pressed to model bank transactions continuously, though even such a wild formal twist may be useful in financial

[8] In calculus the existence of such smooth transitions is supported by the 'Intermediate Value Theorem'.

applications. The decision of whether to go with continuous or discrete time is best left to a case by case approach and particularities of the application at hand.

As significant as the differences between IFS and CFS seem at first, it turns out the two are much closer formally than it first appears, although in logic too, with the exception of temporal logic, the discrete conception of time is the better understood one. There currently remains a fair amount of work to be done on better appreciating the continuous time systems. In the remainder of the paper, we will treat the two together dealing with significant differences as they arise. In contrast to the standard quantitative approach, our approach here is of a global kind. The best known local approach to dynamics, DTL, which we look at next, also uses discreet time.

2.4 Dynamic Topological Logic, DTL

Dynamic Topological Logics as *modal logics* have first been looked at by two Russian-American teams , Artemov, Davoren and Nerode at Cornell [1] and Kremer and Mints at Stanford [8].[9] The two approaches are formally largely identical. They are both based on the simplest IFS, that is, the time considered is discrete, and both use a fairly natural modal language with two temporal and one spatial modality.[10] The main difference–Artemov et al. allow for multiple functions f_i whereas Kremer and Mints treat an IFS with a single function f–turns out not to be of a great formal consequence. We will follow Kremer and Mints's presentation in calling the logic and the system DTL, for Dynamic Topological Logic. The idea for the system is natural. One starts with the simplest IFS, $\mathfrak{X} = (X, f)$. The topological space X of the IFS interprets spatial properties, while the function f determines the temporal behavior of the system. The language chosen to express the properties of X will determine what spatial properties can be expressed, while the language chosen to express the temporal properties will interpret f.

In DTL, one largely concentrates on points $x \in X$ and sets of such points, but one also gains some of the global dynamical perspective by looking at the orbits of a point x. The orbit o_x is a function $o_x : \mathbb{N} \to X$ where $o_x(0) = x$ and for all $n > 0$, $o_x(n) = f^n(x)$, that

[9] The Russian connection may not be entirely coincidental. The Russians have been known to advance the mathematics of dynamical systems in the '40s and '50s when not much interest in such systems existed outside the Soviet Union, at least not in mathematics. It seems that the Western interest in the mathematics of complex systems begins in the late '60s and early '70s as the result of various formal and other scientific advances. Lorenz's 'discovery' of the butterfly effect in the weather systems, Maynard Smith's work in mathematical biology, Mandelbrot's work on fractals, and later on work by Feigenbaum and others in physics of chaos. Though this oft repeated view of history seems overly simplified, there is a definite spike of interest in mathematics of dynamical systems across the disciplinary boundaries in that period. A decently philosophically informed historical summary can be found in say Peter Smith's book *Explaining Chaos* [15].

[10] Plus their existential duals.

is, the result of n applications of f to x. Another way to view o_x is as a countable sequence $\{x, f(x), f^2(x), f^3(x), ...\}$. Essentially, one is interested in general tendencies and behaviors of orbits. Notice that the notion of an orbit extends quite naturally to the continuous time. One simply takes a path along f, that starts with x. In a deterministic CFS such path is clearly unique. o_x is still a function, just that this time the domain is the positive real numbers rather than naturals.

2.5 Modalities and Their Semantics

The three modalities in the language of DTL are $\Box$–the standard topological interior modality of McKinsey and Tarski [11], $\bigcirc$–the temporal next moment modality, and $*$–the temporal henceforth modality. Without going into to much formal detail, here are the semantic renderings of the three modalities:

$\Box\phi$ is true at some point $x \in X$ if there is an open neighborhood U of x, and $\forall y \in U$, ϕ is true at y,

$\bigcirc\phi$ is true at x if ϕ is true at $f(x)$,

$*\phi$ is true at x if for all $n \in \mathbb{N}$, $f^n(x)$ makes ϕ true, where $f^n(x)$ is the result of n successive applications of f to x.

The modalities $\Diamond$–the topological closure operator; and F–'sometime in the future' are also used and of interest, though they are definable as $\Diamond := \neg\Box\neg$ and $F := \neg * \neg$ and thus not needed as primitive.

One could further strengthen the language by adding expressive power to the spatial component of the language. It is well known that for instance topological connectedness cannot be expressed in the language of $\Box/\Diamond$. Furthermore, one could add additional operators to express some metric information contained in the IFS models. The language, however, is already extremely powerful in the sense expressed by measuring computational complexity of the satisfaction problem for the resulting logics.

2.6 Some Computational Properties of DTL and its Fragments

It has been argued that the main advantage of Modal Logics over say their first and second-order counterparts lies in their computational complexity. It is often noted that a modal logic and a first-order logic over the same domain commonly have radically different computational properties. Modal logics, it is said, allow for some quantification while keeping the logic in question not only decidable, but also of a very low complexity.[11] If

[11]For the computational perspective on modal logic see for instance [3]. For a general introduction on computational complexity see [13].

this indeed is the main advantage of Modal Logic over other related counterparts, then DTL does not fare too well. As it has been shown in [6] and [7], DTL becomes undecidable and even not recursively axiomatizable under some very weak assumptions about the time function f. Konev et al. prove a series of high complexity of DTL over some interesting classes of models by connecting DTL with products of modal logics. In other words, they show that DTL has enough expressive power to encode certain class of tilings of a countably infinite grid. Putting these results in perspective, they categorize the full DTL interpreted over the calls of homeomorphisms at a par in computational complexity with the full first-order logic. In fact, most of the interesting fragments of the full language of DTL are also undecidable and thus the best one can hope for are sufficiently tractable axiomatizations. Put strongly, DTL provides us with very little computational reason to use modal languages over say full first-order languages or even full second-order languages in reasoning about dynamical systems. Non recursively axiomatizable modal logics are formally interesting in their own right, of course, and their study has provided us with invaluable insights into reasoning about dynamical systems, but one wonders if there is any way to recover some of the nice computational features of modal logic while keeping with the sprit of DTL's modal view of dynamical systems. We think so, but a change in perspective is needed. One needs to shift from a local to a global perspective. This shift of perspective is a hallmark of dynamical system behavior, as we will argue, and the main interest and power of dynamical system thinking is derived from the features of the global perspective.

2.7 Poincare and Topology of Dynamical Systems

As Poincare has observed in his study of the three body problem, for any given dimension and from a specific global perspective, there are only a relatively small number of kinds of dynamical behaviors that are worth distinguishing. That is, if one looks at the global tendencies of a changing system rather than its local behavior, one can isolate a certain number of points to which the local motion is attracted. He called these points attractors.

Poincare is essentially the father of the study of the theory of complex dynamical systems. In his researches, he was responding to the contest called by the King Oscar II of Sweden to finally solve the problem of calculating the interactions of a set of three heavenly bodies based on their mutual gravitational influence. The two-body variant of the problem was solved by Newton himself. The three-body and the n-body generalization proved a bit of an embarrassment to the mathematical community. The problem went unsolved for about 300 years! Even for the great mind of Poincare, the three-body problem turned out to be hard nut to crack. Having worked on an analytic solution for several years, Poincare concluded that the conceptual apparatus needed to understand such systems was hopelessly tricky. He gave up, but not before he did enough work to win King Oscar's prize[12] and had discovered the topology of attractors of a phase space and classified all possible attractors in one dimensional space, i.e., the real line. He did not, however, solve the three body problem. That had to wait for another dozen or so years, 1912 to be precise. An excellent dissection of various aspects of the three-body problem, from both historical and mathematical perspectives, can be found in the wonderfully subtle discussion of [4].

Poincare's results concerning *attractors* implied that one can say a great deal about the dynamics of a particular complex system with a weak conceptual apparatus that lacks the capacity for detailed local descriptions. Thus, instead of the complete phase space in all its detail, to obtain a reasonably complete dynamical picture, one needs only the information about a small number of distinguished points and their relationships with their neighbors. Equipped with such information one can predict how any particular run of the system will evolve without calculating the details of the trajectory. Here is a particularly simple example that nicely illustrates the global topological perspective introduced by Poincare.

2.7.1 Dynamics of a One-Dimensional System

We consider the following simple single function IFS that we will call RS.

Example 2.8 (IFS$_{RS} = (\mathbb{R}^*, x^2)$) *The space is the set of real numbers $\mathbb{R}$ together with $+\infty, -\infty$. We call this set $\mathbb{R}^*$. The sole 'change' function f is x^2. We stipulate that x^2 behaves as a fixed point on $+\infty$, and sends $-\infty$ to $+\infty$, that is, $f(+\infty)^2 = +\infty$ and $f(-\infty)^2 = +\infty$. Thus for any point $r \in \mathbb{R}^*$, will in the next moment move to $f(r)$, i.e., r^2.*

How will objects in this space behave in the long term? For instance, if the initial condition is 345.65, what will the path look like over many applications of the change function f? What if the initial condition places us at 0.5? Will they converge toward the same point? Poincare has provided us with a general answer to this type of question. Essentially, instead of computing the trajectory of the function x^2 starting with our distinguished points and comparing those trajectories, a tedious task indeed, we can look at the global topology of the system and answer more or less immediately where the trajectories are headed. Poincare's method tells us that there are three distinguished points in this IFS, and those three alone, while ignoring all the uncountably many others, tell us much of what we care to know about this simple dynamics. For instance, we know that a path starting at 345.65 will rapidly tend towards $+\infty$. This follows since 345.65 is a positive number and an

[12]No one lesser than Karl Weierstrass advised the King that Poincare's contribution was substantial enough. This made the fact that there was an actual mistake in the original proposal so much more embarrassing. Poincare later fixed the mistake arriving at the topological accomplishments that we mention here. See [4] for historical details.

increasing infinite sequences of squares of positive numbers will have $+\infty$ as their limits. In fact all orbits with initial states in the set that we will call the *basin of attraction*, $(1, +\infty] \cup (-\infty, -1)$, will converge to $+\infty$.

There is a curious bit of dynamics in the initial behavior of the orbits that start in $(-\infty, -1)$. Before initiating their steady march towards $+\infty$, they first 'jump' into the positive numbers. Except for that initial leap, the two sets of numbers have the same dynamics. This jump, however, is sufficient to ensure that the system described here is not deterministic. It is not stochastic either, but rather it is *over determined.* Every orbit except the one starting with 0 has two distinct starting points. For instance, the two orbits, $(-2, 4, 16, ...)$ and $2, 4, 16, ...)$ are identical but for their starting position. If we think of our IFS as a physical dynamics, the situation is curios indeed. Two rather distant events have exactly the same causal consequences. The problem in this particular IFS is systematic. Every sequence of events that does not begin at the origin has two possible beginnings. Perhaps a theistic accommodationist would find this scenario plausible, but for the rest of us it really shows why we are interested mostly in deterministic and stochastic IFS's. In the example below to which we apply modal languages, we will ensure that this curious causal behavior is ruled out.

The three distinguished points are $0, 1, +\infty$. Each one is a fixed points of f[13], but only two, 0 and $+\infty$, are stable fixed points or attractors. If we were to perturb the initial position away from 0 or $+\infty$ by some small margin ϵ, with enough time, the trajectory of the new starting point would come arbitrarily close to the attractor. The size of ϵ is crucial here. To see this, take for instance the attractor 0. If we perturb the starting position by more than say 1, the new wayward trajectory would tend towards $+\infty$. Any stable attractor has a non negligible *basin of attraction* surrounding it. The basin of attraction of 0 is the open interval $(-1, 1)$. It is called the basin of attraction as every orbit with its starting position inside the basin of attraction of 0 will have 0 as its limit.[14]

$+\infty$ behaves in a way similar to 0. It is a stable attractor. As we said earlier, its basin of attraction is $(1, +\infty] \cup (-\infty, -1)$. Any trajectory with the initial point in this basin will have $+\infty$ as its limit. 1 is also a fixed point, that is, $f(1) = 1$, but its basin of attraction is just a singleton consisting of the point itself. What that means is that any perturbation of the trajectory that begins with the point 1, however small, sends the new trajectory drifting away towards a different attractor. Points that exhibit such unstable behavior are called *repellors*; the flow of the nearby trajectories is diverted away from them.

[13] A point x is a fixed point of a function f, if $f(x) = x$.

[14] This is the standard notion of a limit. Here is a quick informal reminder: Let x be in the basin of attraction of y. Then, for any distance δ however small, there is a natural number n, and every $f^m(x)$ for $m > n$, the distance between $f^m(x)$ and y is smaller than δ. That is, the distance between $f^m(x)$ and y gets smaller and smaller as m increases. Recall that $f^m(x)$ is a shorthand for $f(...(f(x)))$, with f applied m times. So $f^3(x)$ is $fff(x)$.

In one dimensional space with f continuous, there are three kinds of fixed points.[15] Attractors: they attract neighboring orbits from both left and right; repellors: they repel neighboring orbits from both sides, bipolar fixed points: they attract orbits on the right and repel those on the left, or *vice versa*. This is in an important topological sense a complete taxonomy of dynamical behaviors in one dimension. All other points in one dimension can be labelled 'transients'. The kinds of flow one gets in the limited topological arrangement of one dimension are of course limited; the kinds of flow in two dimensions will get more complicated, and further dimensionality will add additional complexity of attractors. In each case, however, there is a limit to the eco-diversity of attractors. We can use the taxonomy of attractors of a space and their interrelations to capture the logic of the space in a modal logical setting.

Before we move one, it is worth noting that the IFS=$(\mathbb{R}^*, x^2)$ is at the same time a CFS, ignoring the slight glitch that the time is defined over $\mathbb{R}^*$ rather than $\mathbb{R}$. The function x^2 is certainly continuous and differentiable, we just need to change our outlook on time. Essentially, the event 2-time units after x would be $f(x+2)$ rather than $ff(x)$ and the orbit that starts at x assuming that $f(r) = x$ and that $r \neq x$ is $[f(r), f(q))$ with q being the least $q' > r$ such that $f(q)$ is a fixed point. The IFS $(\mathbb{R}^*, x^3)$ we introduce below can similarly be transformed into a CFS.

3 A Case Study: IFS=$(\mathbb{R}^*, x^3)$ via Some Qualitative Modal Languages

We now closely examine a deterministic IFS space $RC = (\mathbb{R}^*, x^3)$ and consider some possible modal interpretations over this simple dynamic space.[16] The space is a lot like the IFS $RS = (\mathbb{R}^*, x^2)$ that we examined above, except that it exhibits some further symmetries. The main differences are:

1. The distinguished points of RC are now five fixed points: $0, -1, 1, -\infty, +\infty$ (compared to the three fixed points in RS).

2. Three of the five points, $0, +\infty, -\infty$ are stable attractors, and two, $-1, 1$, are repellors.

The long term dynamic behavior of any object in RC can be approximated from the two facts above as we can readily infer the basins of attraction for the five fixed points. For

[15] This does not hold for general maps. A good example is the Tent Map that we defined in Section 2. It is capable of much more complex behavior, including chaos and hence *strange attractors*.

[16] RC stands for for the real line with the cubing function. This is in contrast to the earlier RS, the same underlying topology with the squaring function capturing change.

completeness of presentation, we list the basins of each point. We label the basin of attraction of a point x, B_x. Then,

$B_{-\infty} = [-\infty, -1), \quad B_{-1} = [-1], \quad B_0 = (-1, 1), \quad B_1 = [1], \quad B_{+\infty} = (1, +\infty]$.[17]

3.0.2 RC and the Local Language of DTL

The first example of a local modal approach to RC is provided by DTL. What makes this approach local is that it does not explicitly account for the attractor level global topological information of the system, at least no mention of such information is made explicitly in the language. If present at all, such global information emerges bottom up from the local detailed description of the model. The topological modalities $\Box$ and $\Diamond$ are interpreted in the standard metric topology over $\mathbb{R}^*$ with the appropriate adjustments to accommodate $+\infty$ and $-\infty$. Further, let $[\phi]$ be the set of points that make ϕ true. Let $\propto A$ stand for 'the largest open subset of A'.[18]

$$[\Box\phi] = \propto [\phi].$$

Let $\star A$ be the smallest closed set that contains A.[19]

$$[\Diamond\phi] = \star[\phi].$$

For any

$$r \in [\phi], \ \sqrt[3]{r} \in [\bigcirc\phi].$$

This follows from the standard definition

$$[\bigcirc\phi] = f^{-1}([\phi]).$$

Finally, the interpretation of $*$ as an infinite conjunction leads to the definition:

$$[*\phi] = \cap_{n\geq 0} \ f^{-n}([\phi]).$$

As we know from the complexity results mentioned earlier, the unrestricted language of DTL is quite potent, and since x^3 is a continuous function, the DTL_{RC} axioms build on

[17] We write [1] for the closed singleton $\{1\}$. Also, it a bit unconventional to think of repellors as having a basin of attraction, but in our view treating their basin as a singleton helps systematize the set of distinguished points, and it facilitates their definition as the fixed points with the singleton basin of attraction.

[18] It is an easy topological observation that this function is i) well-defined, ii) the open subset is unique.

[19] Again, some fiddling with complements and the definition of a closed set shows this set to exist and to be unique.

the axioms for DTL over $\mathbb{R}$ with an arbitrary continuous function. This axiomatization is unknown, but the logic is known for instance to be stronger than DTL over the real plane $\mathbb{R}^2$ with an arbitrary continuous function (the latter logic is known).[20] Finding DTL_{RC} would barely amount to much more than a curiosity, and at any rate, our goal in this paper is philosophical rather than formal. Based on general principles at least, the question of axiomatization seems to be substantially less difficult than the corresponding question for DTL over the reals with unrestricted continuous functions. Answering this question may be an easy, approachable case study in applying modal techniques to a particular dynamical system. We will not concern ourselves here with the following questions, but they are well worth formulating:

Question 3.1 *What is the logic of DTL_{RC}? Is it decidable? Finitely axiomatizable?*

We remark though, that given the simplicity of the function f, and the amount of structure that RC model has, it would not be too surprising if this logic was computationally better behaved that the general DTL over continuous functions.

What does interest us is the issue of the expressive power of this language. DTL has just enough of expressive power that enables it to express some rather structured tilling problems [see [6, 7]], but the approach in such construction assumes that the function f is a homeomorphism. That assumption makes the models of DTL satisfy strong grid interaction properties known as *Church-Rosser* and *Commutativity*, which in turn make the language and logic behave like a class of highly complex modal product languages. Weakening the assumption from homeomorphism to a continuous function simply bars this avenue for assessing the complexity of DTL, but the question of whether this weakening of the assumption actually makes the logic less complex or even perhaps decidable is still–at least as of the time of writing this paper–open. In the case of highly specialized underlying model, however, even if the general case turns out to be undecidable or not even recursively axiomatizable, this special case may turn out to be of low complexity. So what can we say in this complex language? It is often claimed that what you loose on the complexity side of things, you gain in expressivity. Though no real guarantees exist here. You may have an unfortunately designed language that simply expresses all the wrong formal properties. The proofs of high complexity goes through, but no interesting new properties become definable. In our case, the question is what interesting properties of dynamical systems are expressible? Mints and Kremer show that this propositional language can express some interesting topological theorems [see [9], sec. 3].

In the systems based on $\mathbb{R}$, we would minimally like to be able to express that a point is fixed, a repellor, and attractor respectively. Fixed points are defined as points that validate $p \to \bigcirc p$ or equivalently $p \to *p$. It turns out, however, that when one tries to extend this reasoning further, even with the assumption that the function f is continuous and the

[20]See [9] for up to date account of the state of DTL

rather strong assumption that the space is $\mathbb{R}$, one runs into problems trying to define even the obvious global properties like being an attractor or repellor. In a longer paper, we would define a notion of bisimulation and actually prove that the the two properties are undefinable, but here we just observe that the two are not definable in any obvious way.[21]

There are several ways to extend the DTL language while preserving the local perspective. Most extensions pertain to the spatial fragment, but there are also interesting temporal variants. Among the spatial extensions, adding a universal modality that enables one to express *topological connectivity* and adding some amount of expressivity over the metric properties count as the most obvious. The main temporal proposal would base the logic on an CFS, a continuous time based dynamical system. This system would presumably have only one modality as the next operator, $\bigcirc$ does not make sense here. Related is an exploration of $\Box, *$ fragments of the logic. The $\Box, \bigcirc$ fragment has been studied extensively, but little or no attention is paid to the other obvious option, $\Box, *$.

3.1 Qualitative Modal Operators

3.1.1 A Simple Global Language for RC

Whatever the actual complexity of DTL_{RC} turns out to be, the fact that DTL plays an improtant role in the spectrum of modal dynamic logics is undeniable, as is the fact that it is a langueage that is detail oriented and perhaps best suited to the applications where a great deal of precision is needed and where one readily sacrifices computational efficiency for the extra added detail. The language introduced in this section is on the opposite side of the spectrum. The high level topological global language is best suited for understanding the rough global topological structure of the dynamics embodied in RC.

The language is so weak that it does not need the full detail of the RC's IFS. Instead, we obtain a finite model by filtering through most of the points out of RC. We preserve all the fixed points, as well as the barest outlines of their basins of attraction. Everything else is disposed of. We call the procedure of shrinking the size of the IFS, *attractor filtering.*

The *attractor filtering* of an IFS X is procedure designed to collapse large often uncountable sets of points that form the IFS into more manageable, in fact often finite set. The goal is to preserve as much of the global dynamics of the IFS as possible while eliminating all

[21]Tamar Lando found a curious class of models that defeat all reasonable attempts at defining these two properties. One of the models consists of a countable sequence of points approaching 0 from the right, each of the points is an attractor for some sequence of the form $f(x), f^2(x), f^3(x)$, but no such sequence approaches 0 itself. The model seems to be indistinguishable in the language of DTL from either the model that has 0 as an attractor, or conversely a model that has 0 as a repellor. So curiously, although DTL has explosive complexity, it makes it difficult to define even the simplest of global dynamic properties.

the extraneous detail. We call the new collapsed frame AF_X. $AF_X = (Y, R)$ is a pair consisting of the underlying set Y and a set of relations R. We define AF_X in two stages. First we define the set Y.

Definition 3.2 *Given an IFS X, we obtain the set Y_X by* attractor filtering *of X.*

i) For every fixed point r in X, we add y_r to Y_X.

ii) For each fixed point r, we add up to two new points to Y_X. If there is an orbit o_x for $x < r$, and r is the limit of o_x, we add a point o_L to Y_X. Similarly, if there is an orbit o_x for $r < x$, with r as limit, we add a point o_R to Y_X.

iii) No other points are added to Y_X.

Thus, for RC, the attractor filtering, Y_{RC}, consists of the following nine points. For simplicity of exposition we let i abbreviate $-\infty$ and j abbreviate $+\infty$.

$$y_i,\ o_R^i,\ y_{-1},\ o_L^0,\ y_0,\ o_R^0,\ y_1,\ o_L^j,\ y_j$$

Next, we need to choose a relation set appropriate to the dynamics we wish to capture. This is admittedly a harder task. In the simple model *RC* we can get away with adding just two relation, R_A and R_R to capture the simple dynamics. To understand what these relations do, we need to look at set Y_X. In some sense, Y_X gets its significance by mixing points and orbits. For example, o_L^0 is clearly a representative of a class of orbits that approach 0 from the left. Now, what about y_0? Is it an orbit consisting of all 0s or just a point? We in fact don't need to answer this question. It is in a sense akin to the relation between light and wave/particle dichotomy. Thus, we treat y_0 as both a representative of an orbit, and a point that attracts other orbits. The relation R_A holds between two points z, w if in the original model the sequence z is attracted to w. Thus, since any sequence that starts in $(-1, 0)$ is attracted to 0, we say that o_L^0 which represents all such sequences is related to y_0, $R_A(y_0, o_L^0)$. Similarly, $R_A(y_0, o_R^0)$, $R_A(y_i, o_R^i)$, $R_A(y_j, o_L^j)$, and slightly less obviously, $R_A(y_i, y_i)$, $R_A(y_{-1}, y_{-1})$, $R_A(y_0, y_0)$, $R_A(y_1, y_1)$, and $R_A(y_j, y_j)$.

The second relation, R_R, records the pairs of a sequence and a point where the sequence drifts away from the point. Thus since all orbits starting in either $[-\infty, -1)$ or $(-1, 0)$ are repelled away from -1, both $R_R(y_{-1}, o_L^0)$ and $R_R(y_{-1}, o_R^i)$ hold. Furthermore, $R_R(y_1, o_R^0)$, and $R_R(y_1, o_L^j)$ also hold. Bellow is the graphical representation of the attracting and repelling in the model. The arrow with the tip on the lower side represents attracting, and the relation R_A is the inverse of the arrow in the sense that $R_A xy$ iff $y \rightharpoondown x$. Similarly the arrow with the upper tip represents repelling, but this time the relation is as it is, not an inverse: $R_R xy$ iff $x \rightharpoonup y$.

$$y_i \leftharpoondown o_R^i \leftharpoonup y_{-1} \rightharpoonup o_L^0 \rightharpoondown y_0 \leftharpoondown o_R^0 \leftharpoonup y_1 \rightharpoonup o_L^j \rightharpoondown y_j$$

As far as the language goes, we predictably add the pair of modalities $\Box_A$ and $\Box_R$. To enable us to talk readily about the dynamical system globally and express claims like 'all fixed points are attractors,' and other related propositions about the overall behavior of the system, we add the global modality U. Semantically $\Box_A$ and $\Box_R$ are standard modalities interpreted via their corresponding relations R_A and R_R:

$\Box_A \phi$ is true at a point x, if all points y such that $R_A xy$ make ϕ true;

$\Box_R \phi$ is true at a point x, if all points y such that $R_R xy$ make ϕ true.

The modality U is not dependent on a relation,

$U\phi$ is true at x if every point y makes ϕ true.

Each modality has an associated existential variant: $\Diamond_A, \Diamond_R$, and E, all defined in the obvious way.

Notice that repellors and attractors are now defined as a simple matter of accounting. For instance, saying that a point has three R_A successors in one-dimensional model will ensure that it is an attractor. For end points, two attractors and, for the left end point, no relata along either relation to the left. Similar story goes for the right end point, and attractor in general is then a disjunction. Repellors are defined simply by saying that an attractor has a repellor relatum.

This just is the simplest one dimensional example of a dynamical system and the taxonomy of attractors is relatively simple. As we mention briefly above, there really are only three different kinds of fixed points, and the dynamical systems are the ones that can be assembled by combining such attractors on a line, not much to report really. It is still however somewhat surprising that,

Proposition 3.3 *Every one-dimensional IFS with a finite number of fixed points is decidable in the three-modal language of AF.*

The claim follows from the fact that the filtered model will contain only finitely many points. Extending to the general case for all one-dimensional spaces would likely not be more difficult.

3.2 Modal Languages for Higher Dimensional Dynamical Systems

This however does not make the approach trivial. As the number of dimensions increases, the complexity of the behaviors that attractors are capable of increases rapidly. Already in two dimension, we have *saddle nodes*, nodes which attract orbits along one dimension of approach and repel orbits along the other dimension. Furthermore, in two dimensions, the behavior is further complicated as the result of the fact that we now not only have fixed points, which endlessly repeat a single point, but we now have *periodic orbits* that circle periodically along some finite sequence of points. Furthermore, such periodic orbits themselves can be repellors or attractors. One can additionally have circles on the plane that serve as attractors or repellors, and they can be both at once, say repelling in the interior of the circle and attracting on the exterior. Then there are *quasi periodic* behaviors where periodic behavior is not quite achieved, but points periodically remain close enough to the original points of period. For example, an orbit of quasi period 3 may have the sequence $3.4, 6.7, 54.2, 3.6, 6.5, 54.9, 3.2.6.6, 54.5, ...$ Thus, although the orbit does not return to the exact starting point after three iterations, it remains close enough to the starting point every three iterations, and close enough to the second point in the sequence on 2nd, 5th, 8th,... period. Another way to see this, is by noticing that though not exaclty periodic, if one truncates enough decimal places, one ends up with a periodic sequence: $3, 6, 54, 3, 5, 54, 3, 6, 54, ...$

Such more complex behaviors would clearly require a more sophisticated modal language to capture interesting phenomena in their global behavior, and in fact as pure repellors and attractors are relatively rare, our language would not be of much use, but a language in its spirit, where the IFS is filtered resulting in a small set of distinguished points. The remaining points are then related via a set of relations most of which essentially have to do with repelling and attracting orbits and the points, or in this case sets of points, that do the attracting and repelling.

There are even more complex behaviors with such obscure labels as *riddled basins*, *fractal boundaries*. Riddled basins are areas around a point that contain intertwined attracted and repelled regions, wheres fractal boundaries are boundaries of a basin of attraction that have fractal properties such as nondifferentiablility and fractal dimension. It seems like an interesting challenge to devise a small set of modal operators capable of expressing various of these topological properties of dynamical system and perhaps introduce a modal logical classification of kinds of dynamics.[22]

So far we have seen the most local class of modal logics of dynamical systems, DTL, and the most global one, based on the attractor filtering of IFS. There are a variety of options in the middle. One can for instance weaken the language of DTL and explore the properties of such weakened DTL system. Completeness and decidability of some such fragments have been explored by Kremer, Mints and others. It would also be interesting to see what are some of expressive features of such languages. Further, one can turn the complexity argument on its head and argue that since DTL has computational properties of such high complexity, why not try strenghten the language to add extra expressive power with the only restriction that the computational properties of the extensions be no worse than those of the systems they started with. So one starts with the class of axiomatizable DTL models, perhaps any extension that preserves axiomatizability is a fair game. There is a variety

[22] A great source on dynamical systems in general, and variety of global behavior is [17].

of desirable spatial properties that one may think of adding. From the simplest like the universal modalities U and E, to the more exotic extension towards stronger quantification or metric, and various geometric languages.

Then on the temporal side, one may wish to see what would happened if the IFS were to be replaced by CFS, or continuum based time. It goes without much argument that the high complexity of time in DTL and its interaction with space give result in the high complexity of the system. The fragment of the language that involves $\Box$ and $\bigcirc$ only has been looked at extensively. Curiously, however, the fragment that only involves $\Box$ and $*$ has not plaid a major role in DTL community. Some properties of such combination are already known from the current author's work on the products of topological modal logics, and some extensions of that work by Kremer[23].

Question 3.4 (Some research questions) *Adding the universal modality U to DTL. Does the universal modality increase the complexity of DTL over the class of homeomorphisms (continuous functions)? Does it simplify or further complicate the axiomatizations, when applicable? Zacharyaschev et al. have proposed some interesting modal metric languages. Same concerns as above for such language extensions.*

What is the general logic for the $\Box/*$ fragment of DTL over the class of homeomorphisms (continuous functions)? What are computational properties of this fragment?

Let $\bigoplus$ be a single temporal modality interpreted over the class of CFS with a single time function f. What is the most plausible semantic interpretation for $\bigoplus$? For instance, is it 'all moments hereafter' or 'all moments in an interval starting with the current point' or something entirely different? Let's call such logic DCTL for Dynamic Continuous-time Temporal Logic. What is the logic of DCTL over the class of homeomorphisms (continuous functions)? What is the complexity?

We are moving on now from the more extreme ends of the spectrum, a fine-grained local perspective of DTL, and global pattern based perspective of AF, towards logics that combine aspects of both global pattern based thinking and honest labour of detailed approach.

3.3 Dynamic Epistemic Logic and the IFS Perspective

If one were to simply poll the number of researchers in the field, the Dynamic Epistemic Logics (DEL) form by far the most important and most studied class of dynamic logics. Not only has it yielded some of the most interesting technical questions in the field, but

[23] The logic of $*$ in the most standard case just is $S4$, and so the DTL model that only involves $\Box$ and $*$ is a product of an Alexandroff topology and an arbitrary topology. The complexity and various axiomatizations are known for a number of classes of topologies such as Alexandroff and Alexandroff, Alexandroff and Metric, Alexandroff and singleton $\mathbb{Q}$, etc.

it has also sprung the most richly diverse class of enhancements and offshoots–logic of action, probabilistic dynamic logic (see the Introduction and Chapters 1–5, this volume), the enhancement with preference and other modalities (Chapter 5, this volume), the belief variant, to name just a few. Furthermore, it is probably the best philosophically motivated logic in the dynamic realm. What can the IFS perspective say about this class of logics? Curiously, although there does not seem to be a clear connection between the IFS and DEL, we have started thinking about the IFS perspective while looking at some metalogical problems in Public Announcement Logic in DEL paradigm. DEL is usually presented as a system that starts with a standard modal epistemic logic. Normally one is already in a multimodal setting, that is there are n agents interacting epistemically. We will concentrate here on the Public Announcement subclass of DEL. In such models, change in the system is then introduced in form of an externally made public announcement, and the subsequent update that the announcement forces upon the knowledge of the agents in the model. Agent knowledge, as in the standard multi-agent epistemic logic, is represented as a Kripke frame that is either transitive and reflexive (S4) or those two plus symmetric (S5). The multi-agent system is formed by having relations $R_1, ..., R_n$ over a single universe, with each relation representing another agent. An announcement then changes the model, and the change in the model comes to represent the change in what the agents know. How different or similar is this DEL set-up from the IFS approach? Here are the main points of similarity/difference:

3.4 DEL vs. IFS

Space: 1. By taking all sets $U_x = \{y \mid Rxy\}$ as a base of a topology, one can show that (S4), and (S5) frames are readily seen as topologies. Thus, like in IFS, the base space is a topological space.

2. The topology induced by (S4)-frames is called *Alexandroff* topology. (S5)-frames induce *Almost Discrete* topology over their underlying space.[24] Unlike the topological element of IFS, these topologies are not necessarily metrizable, and they are certainly not the standard Euclidean metric topologies that we have been using in the examples.[25]

3. IFS was defined as having single topology over its underlying space. The multi-agent DEL has multiple topologies.

Time: 1. We insisted that IFS has at least one function representing change, possibly

[24] A topological space is *Almost Discrete* if every open set is closed.

[25] To see that that, for instance, Almost Discrete Topology is not metrizable, notice that its basic open sets induce a partition of the space exactly as R does in (S5)-frames. The elements of the same partition will not be metrically distinguishable. Any metric would have $d(x, x) = 0$, but for any other y in the same partition as x, $d(x, y) > 0$ by the definition of a metric. In a finite partition, which can not be ruled out, this would make singleton x open which it is not by the fact that it is part of the partition.

many different ones. Thus, whatever the notion of change happens to be in DEL, there is no reason why it should not be representable in an IFS.

2. On the standard view, the change in DEL literally removes a subset of the DEL model. The change functions f in IFS leaves the underlying space intact.

3. DEL has a designated point, the real world. Let x be the designated world. The announcements restrict the class of functions to the ones where for every n, there is a y, s.t. $f^n(y) = x$. That is, the real world has to survive each update; it has to be in the range of every updating function. This is another way of saying that the announcements have to be truthful.

How important are these differences? Let us see what an IFS_{DEL} looks like before we set out to compare them.

Definition 3.5 **(IFS_{DEL})** *As before we begin with some set of points X. We now, however, need to designate a distinguished point $g \in X$.*[26]

i) Let $\mathcal{O} = \mathbb{O}_1, ..., \mathbb{O}_l$ be a set of Alexandroff [Almost Discrete] topologies over X.

ii) Let $\mathbb{F} = f_1, ..., f_k$ be a set of functions $f_i : X \to X$ that satisfy the following restriction: for each i and all $n \in \mathbb{N}$, $f^n(g)$ is in the range of f^n

Then, $IFN_{DEL} = (\mathcal{O}, \mathbb{F})$.

As a first observation concerning the difference between DEL and IFS, note that we can simply treat DEL and PAL as an argument for expanding the concept of IFS to allow for multi-topology. We already allow multiple time/change functions f_i, so unless there is an independent reason to discriminate against topological multiplicity, it seems like a reasonable accommodation. Moreover, on the issue of metricity of the topologies, we can go either way. We can either argue that requiring the topologies of DEL to be metric makes DEL more like the standard dynamical systems, and, hence, may lead to transfer of useful results from the theory of dynamical systems to DEL. One would be especially tempted to take this topological route with respect to DEL, if one could show that none of the usual theorems about properties of DEL are altered significantly by the new restriction that the spaces be metric rather than the usual (S4) and (S5). One could, conversely, argue that the notion of space in IFS needs to be liberalized, not just for the sake of DEL, but also for the sake of other spaces that we mention together with the definition of IFS above. Although we do have some partiality towards the topological perspective, it is not strong enough to push us to either side of this dilemma. So, take your pick.

On the temporal side, we are, at least initially, interested in restricting functions in $\mathbb{F}$ to the class of update functions. The updates are in the language of multi-agent epistemic logic and thus we will have functions corresponding to atoms, the booleans, epistemic modalities,

[26] As in Kripke, g is chosen for *Gaia*, mother earth, or real world.

and perhaps some of the group modalities such as group relative Common Knowledge and Universal Knowledge modalities. We can now set the required restrictions on our update functions. For instance, standardly, for a propositional variable p, the function f_p, that is, the update based on the announcement that p holds (letting $[p]$ stand for the set of points making p true),

$ran(f_p) = [p]$[27] is the golden standard of Public Announcement Logic. After an announcement of p, p becomes true everywhere in the updated model. It also becomes common knowledge among all the agents at each remaining point. The IFS approach strongly suggests, however, a variety of other ways to update an atom. The obvious weakening would be $ran(f_p) \subseteq [p]$. This weakening would go well with some global restriction on all f_is, say, that all functions have their ranges be an open set. There are other plausible update restrictions. Why, for instance, do we want to insist that no $\neg p$ points survive the p-update? We could instead insist, say, that the probability of $\neg p$ after the update is 0, but that alone by no means entails that not $\neg p$ states survive. For instance, we could insist that $ran(f_p) \cap [\neg p]$ has measure 0. That means that the probability of $\neg p$ given $ran(f_p)$ would still be 0, but there could be as many as a countable infinity of $\neg p$ points there, provided that they are sparsely distributed, that is, the set of such points has no density. One can even have uncountably many point if the remaining set of $\neg p$ points is say the classical Cantor set over $[0, 1]$ contrasted with the rest of the interval $[0.1]$. The options are literally countless.

There is no good reason to stop here. We can rule out updates that leave more than $1\%, 3\%, 5\%, 10\%, 25\%$ of the $\neg p$ points. As a matter of fact, if I could have my in-class public announcements be followed by 80% of my students, I would count it a success. Whether such restrictions would ultimately work out in the setting of PAL, depends on a lot more than just defining f_p, but the IFS perspective at least puts a plethora of options to a PAL/DEL researcher to explore.

As it ought to be quite familiar to a DEL action community, further restrictions can be devised in interactions among update functions for various formulas. For instance, it seems plausible that $ran(f_{\Box_i p}) \subseteq ran(f_p)$ should hold if $\Box_i$ interprets topological interior operator. As a homework exercise, what would for instance the restriction $ran(f_{\Box_i p}) = ran(f_p)$ tell us about our set $\mathbb{F}$ if we also insisted that each update give us the biggest possible range?

In fact, as we know from the dual language of DEL that allows for the talk of actions, the update functions for various sentences of the epistemic language have to cohere in a certain sense, and it is also important that they sequentially compose in a plausible manner. Thus one needs a set of principles about pairwise combining of function that produce such a coherent picture. For instance, it would be foolish to require that $ran(f_p) = [p]$ and that $ran(f_{\Box_i p}) = ran(f_p)$, at least if one does not desire to interpret all propositional variables

[27] $ran(f)$ stands the for the range of f and $dom(f)$ stands for its domain.

as open sets. In fact, if a pair of restrictions like this are executable, then $[p]$ is an open set. But even if $[p]$ were an open set to begin with, there is not guarantee that some intervening set of updates has not destroyed this property. Thus we arrive to the achetypal DEL question: can the update f_ϕ be carried out at this time?

What this question suggests is that our set $\mathbb{F}$ of IFS_{DEL} is under specified. We need to decide what happens when a f_ϕ does not meet its prescribed restrictions. For instance one announces that $\Box_i p$ and the pair of restrictions $ran(f_p) = [p]$ and $ran(f_{\Box_i p}) = ran(f_p)$ both hold. Do we just say that the preconditions for this announcement are not met if p is not an open set? Or do we perhaps update and throw out one of the restrictions? The standard PAL way is not to update. This is akin to observing that f_ϕ has crashed after 0 applications. Similarly, if ϕ does not hold at the designated point g, f_ϕ crashes immediately, but this could change with a lucky sequence of updates. Thus both, there may be updates that you can make now, or perhaps even make for some finite number of times, which then cease to be updateable. Conversely, there could be an update that requires some finite other set of updates before it can be made. This gives us a classification of kinds of update functions f_ϕ:

There are f_ϕ that can:

i) never be executed in some model M. ($\phi = p \wedge \neg p$)

ii) not be executed now, but there is a sequence of updates $\langle f_1, ..., f_n \rangle$ of size n, that when executed enables us to execute f_ϕ. ($\phi = \Box_i p$ that is false now at g, but becomes true with some updates.)

iii. be executed once, and never again. Twice and never again, thrice and never again, ... (Once, Moore's formula.)

iv. [be executed some finite number of times, then again several times but not before another sequence has been executed, ...] (not sure about these)

v. always be executed infinitely many times, alone, or in combination with other sequences, whatever the circumstances. (Propositional variables true at g are like this.)

We can now turn the questions, like the well known one about which formulas are preserved after updates, to questions like: which update functions never change applicability of other update functions? Which update functions are self undermining? What exact features make them so? Which functions undermine other functions? Which ones? For instance, a $f_{\neg p}$ update undermines all existential p updates that even if they were executable before this one was made, are not executable afterwards. Thus, one can build a tree of possible executions to try to understand various dependencies among update functions.

3.4.1 DEL, Global or Local Logic?

DEL has a curious status with regards to the question of at what level it reasons about its dynamics. On the face of it, it is local system as it only looks at points and the relations between them, but when one asks the question of where the agents are in the model, one realizes that agents are nowhere to be seen. They certainly are not associated with any given point, as any point except g can disappear without anyone even skipping beat. They are not in the connections either, at least not in any simple sense. Rather, agents and their epistemic properties *emerge* from from the structure of the model. Now if the emergence is the hallmark of the global perspective, then DEL model finely exhibits the global approach.

4 Conclusion

We hope to have demonstrated that IFS provides not only a good global system for comparing various systems of Dynamic Modal Logic, but also that thinking about dynamics in this way raises a variety of new and interesting questions. The hope for the future is to look at some of the proposed avenues in greater detail.

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